# Research Statement

**Justin Sawon**

## 1 Overview

My research area is complex algebraic and differential geometry, incorporating ideas from topology and mathematical physics. The unifying theme has been hyperkähler geometry, a kind of geometry based on the quaternions, but approachable from many different directions.

- In my doctoral and early work I studied curvature invariants of hyperkähler manifolds arising from the Rozansky-Witten sigma model [31]. I also worked on the 3-dimensional TQFT and the knot invariants arising from Rozansky-Witten theory.
- Inspired by O'Grady's discovery [22] of a new hyperkähler manifold in 1999, I pivoted to consider more algebro-geometric problems. In particular, I was intrigued by his use of Lagrangian fibrations and started a long (and ongoing) investigation into their properties.
- A large part of my work has involved derived categories, Fourier-Mukai transforms, and Bridgeland stability conditions, essential tools in modern algebraic geometry. I usually maintain an eye toward geometric applications and connections to Mirror Symmetry.
- Other topics I have considered include generalized geometry (à la Hitchin), its relation to Mirror Symmetry, twistor spaces, and applications of gerbes in arithmetic geometry
- Recently I have renewed my investigation of the topology of hyperkähler manifolds using Rozansky-Witten invariants, and I am exploring a new direction of Eynard-Orantin topological recursion and its application to Hitchin systems and Lagrangian fibrations.

## 2 Rozansky-Witten invariants

Rozansky and Witten [31] described a sigma model based on maps from a three-dimensional manifold to a hyperkähler manifold, yielding invariants of both. A perturbative expansion of the path integral leads to invariants $b_\Gamma(X)$ of the hyperkähler manifold $X$ indexed by trivalent graphs. In my thesis and later work [32, 33] (including with Hitchin [12]) I proved:

- Every Chern number of $X$ arises as a Rozansky-Witten invariants $b_\Gamma(X)$, for a suitable choice of graph $\Gamma$, but there are more Rozansky-Witten invariants than Chern numbers.
- Relations in graph cohomology produce a relation

  $$\frac{1}{(192\pi^2 n)^n}\frac{\|R\|^{2n}}{(\mathrm{vol}M)^{n-1}} = \sqrt{\hat{A}}[M]$$

  between the $\mathcal{L}^2$-norm $\|R\|$ of the curvature of $X$ and the characteristic number coming from the square-root of the $\hat{A}$-polynomial. In particular, the latter must be positive.
- Rozansky-Witten invariants can be generalized to produce invariants of vector bundles on $X$.

With Roberts [30], I considered other generalizations of Rozansky-Witten invariants: to non-compact manifolds and quaternion-Kähler manifolds.

As for the quantum invariants of three-manifolds, in [34] I gave a mathematical construction of the associated topological quantum field theory, which is similar to the Chern-Simons TQFT.

# 3 Lagrangian fibrations

A *Lagrangian fibration* on a hyperkähler manifold $X$ is a surjective holomorphic map $X \to \mathbb{P}^n$ whose general fibres are abelian varieties that are Lagrangian with respect to the holomorphic symplectic form. In [35] I proposed a program to use Lagrangian fibrations to study and potential classify hyperkähler manifolds, and the challenges in this program have guided my work in the subsequent decades. Some of my main results include:

- The Hyperkähler SYZ Conjecture: The Hilbert scheme $\mathrm{Hilb}^n S$ of a K3 surface admits a Lagrangian fibration if $S$ contains a primitive curve with self-intersection $2(n-1)k^2$ for some integer $k$ [37]. (This existence result was later completed by Bayer and Macrì [1], who gave necessary *and* sufficient conditions.)
- If $\Delta \subset \mathbb{P}^n$ is the hypersurface parametrizing singular fibres then [38]

$$\deg\Delta = 24\left(n!\sqrt{\hat{A}}[X]\right)^{\frac{1}{n}}.$$

- An *isotrivial* fibration is one whose smooth fibres are all isomorphic to a fixed abelian variety. In [41] I classified isotrivial elliptic K3 surfaces and used them to construct new isotrivial Lagrangian fibrations on *orbifolds* in higher dimensions.
- In [42] I conjectured that a Lagrangian fibration by Jacobians of curves must be a Beauville-Mukai system [21, 2], i.e., the curves must all lie in a single K3 surface. I proved this under the hypothesis that the degree of the discriminant locus $\Delta$ is greater than $4n + 20$. Later I proved this for genus $n = 3$, 4, and 5 curves [43] with requiring the hypothesis on $\deg\Delta$.
- If we fix the polarization type of the abelian fibres (and impose several other natural hypotheses) then there are only finitely many connected families of Lagrangian fibrations [44].

## 3.1 Recent work

Proceeding on from my classification results for Lagrangian fibrations by Jacobians, I am now working on Lagrangian fibrations by Prym varieties; some of the main directions are outlined in [46].

Together with my student Chen Shen, I constructed a new example of a Lagrangian fibration on an orbifold. Though we followed ideas of Markushevich-Tikhomirov [18] and others, we also found an alternative description of the variety which enabled us to prove that it is a *primitive symplectic variety*.

**Theorem 3.1 ([51, 52])** *There exists a Lagrangian fibration on a primitive symplectic variety of dimension 6 with fibres* $(1, 2, 2)$*-polarized Prym varieties. It is constructed from a K3 double cover of a degree one del Pezzo surface, with fibres the Prym varieties of genus 5 curves covering genus 2 curves.*

I have identified the dual fibration, which comes from Prym varieties of *reducible* curves on *reducible* surfaces, suggesting an unexpected method for constructing more new Lagrangian fibrations.

Donagi-Ein-Lazarsfeld [7] showed that the Beauville-Mukai system (with Jacobian fibres) deforms to a compactification of the GL-Hitchin system [11]. Other Hitchin systems have Prym varieties as fibres; with my student Chen Shen, I extended their deformation result in the following way.

**Theorem 3.2 ([50, 52])** *A compact "K3-del Pezzo" system (such as the fibration by Prym varieties in the theorem above) can be deformed to a compactification of the* Sp*-Hitchin system.*

The significance of this result is that it connects non-compact hyperkähler manifolds (i.e., Hitchin systems, well-studied via differential geometry) to compact hyperkähler manifolds (usually studied via algebraic geometry), opening up many new avenues to explore. Based on these deformations, analogies between compact and non-compact Lagrangian fibrations, and also duality between fibrations (analogous to geometric Langlands in the case of Hitchin systems), I proposed a way of organizing examples of Lagrangian fibrations [46]. My proposal provides both a broad framework in which to organize Lagrangian fibrations and suggests where to look for new examples. The ultimate goal is a classification of Lagrangian fibrations, something that seems feasible at least in low dimensions.

Some related current projects include:

- I am calculating invariants of dual fibrations in order to prove Mirror Symmetry relations at the cohomological level. For example, Debarre's fibration [5] on generalized Kummer varieties has non-principally polarized fibres. By imitating the duality between SL and PGL-Hitchin systems due to Hausel and Thaddeus [10], I constructed the dual of Debarre's fibration [46]. In certain dimensions I can calculate the stringy Hodge numbers of the dual, which is an orbifold, and verify that they agree with the Hodge numbers of Debarre's fibration, as predicted by Mirror Symmetry.

- Isotrivial Lagrangian fibrations on generalized Kummer varieties were described in [41]. I can also construct their duals, and calculate their stringy Hodge numbers in certain dimensions. This once again verifies a Mirror Symmetry relation. I aim to extend this to arbitrary dual fibrations.

- With my postdoc Xuqiang Qin, I aim to classify Lagrangian fibrations in dimension four. Markushevich proved that the only examples with principally polarized fibres are Beauville-Mukai systems. We can show that the only examples with fibres of polarization type $(1,2)$ are Markushevich-Tikhomirov systems [29]. Conjecturally, examples with fibres of polarization type $(1,3)$ must be Debarre systems, and other polarization types cannot arise at all.

- With my student Paul Teszler, I am exploring consequences of the P=W conjecture of de Cataldo, Hausel, and Migliorini [6]. This result relates the topology of a fibration to its Hodge structure in an unexpected way, and was recently proved independently by Chen and Maulik and by Hausel et al. Our goal is to use the P=W conjecture and the Decomposition Theorem to calculate invariants of some of these Lagrangian fibrations by Prym varieties.

I also continue to study the structure of general Lagrangian fibrations. Just as a general elliptic K3 surface will have only nodal rational curves as singular fibres, I conjectured that a general Lagrangian fibration will have only semi-stable singular fibres in codimension one, and gave a partial proof of this in [47]. This result complements work of Hwang and Oguiso [15], who gave a Kodaira-type classification of singular fibres in codimension one.

## 4 Derived algebraic geometry

Fourier-Mukai transforms [20] between derived categories of coherent sheaves on varieties have become an essential tool for studying moduli spaces of sheaves. Many examples of birational morphisms and isomorphisms between moduli spaces are induced by equivalences of categories. I have used Fourier-Mukai transforms in various ways:

- In [36] I studied relative Fourier-Mukai transforms between dual Lagrangian fibrations. A 'Tate-Shafarevich twist' of one fibration corresponds to introducing a gerbe $\beta$ on the dual fibration, and the equivalence becomes one of Căldăraru's twisted Fourier-Mukai transforms [4],

$$\Phi : D^b(X) \longrightarrow D^b(\check{X}, \beta),$$

provided one can extend the equivalence over singular fibres.

- I proved that the above equivalence extends over singular curves for the genus two Beauville-Mukai system [39]. The geometric application was to relating torsors over a Lagrangian fibration: by calculating the space of gerbes arising above I showed that there is a continuous family of Lagrangian fibrations deforming between the original fibration and its Tate-Shafarevich twist.
- My proof of the Hyperkähler SYZ Conjecture for Hilbert schemes of points on K3 surfaces [37] (i.e., conditions for the existence of a Lagrangian fibration) used a Fourier-Mukai transform to identify the Hilbert scheme with a Beauville-Mukai system.
- In [40] I constructed deformations of Fourier-Mukai transforms under gerby and non-commutative deformations of K3 surfaces.

A lot of this work is inspired by and intended to help comprehend Mirror Symmetry: after a hyperkähler rotation a holomorphic Lagrangian fibration becomes a special Lagrangian fibration, as arises in the Strominger-Yau-Zaslow (SYZ) conjecture [53], and the derived equivalences can be viewed as manifestations of Kontsevich's Homological Mirror Symmetry [16].

## 4.1 Recent work

Bayer and Macrì [1] contains an extensive investigation of Bridgeland stability conditions on K3 surfaces, modulo space of stable objects, and applications to the Minimal Model Program (see [45] for a brief summary). The central idea is that the space of Bridgeland stability conditions admits a wall-and-chamber structure, and crossing a wall induces a birational modification of the moduli space which can be described explicitly. These ideas have been extended to threefolds by many authors.

Let $S \subset Y$ be a K3 surface contained in a Fano threefold, and let $\mathcal{M}_S$ and $\mathcal{M}_Y$ be moduli spaces of stable sheaves on $S$ and $Y$. Restricting sheaves to $S$ gives a map $\mathcal{M}_Y \dashrightarrow \mathcal{M}_S$. Tyurin showed that this is often a Lagrangian embedding (see Beauville [3]). With my student Paul Kruse, I investigated what happens to this embedding when we vary the Bridgeland stability conditions for the moduli spaces. Wall-crossing in the space of Bridgeland stability conditions induces birational modifications of the moduli spaces along Brill-Noether loci, and we completely described the interaction of these modifications of $\mathcal{M}_Y$ and $\mathcal{M}_S$ for some examples on a quartic K3 surface $S$ in $\mathbb{P}^3$ [17]. We are now using this theory to find new Lagrangians (aka "branes") in the hyperkähler manifolds $\mathcal{M}_S$.

My postdoc Xuqiang Qin has studied moduli spaces of instantons on Fano threefolds [23, 24] and identified them with moduli of Bridgeland stable objects [25]. With Qin, I am using the ideas above to better understand moduli of stable objects on Fano threefolds. In particular, we are finding 5-dimensional spaces $\mathcal{M}_Y$ that embed as Lagrangians in O'Grady's 10-dimensional hyperkähler manifold [22]. An interesting new feature is that we can fix the K3 surface $S$ and vary $Y$ (unlike with $\mathbb{P}^3$), allowing us to study deformations of Lagrangians in O'Grady's space.

Variation of stability conditions has also been effective in describing all minimal models of moduli spaces. Hellmann described all minimal models of the Hilbert scheme $\mathrm{Hilb}^5 S$ of a general degree two K3 surface; this case is interesting because $\mathrm{Hilb}^5 S$ is birational to a Beauville-Mukai system [37]. In a series of papers [26, 27, 28] we have considered *all* Hilbert schemes $\mathrm{Hilb}^n S$ that are birational to Beauville-Mukai systems, and developed algorithms to compute their minimal models. For instance:

**Theorem 4.1 ([26])** *The Hilbert scheme* $\mathrm{Hilb}^{10} S$ *of a general degree two K3 surface has eleven birational models, each a moduli space of stable objects for different Bridgeland stability conditions. They are related by flops in Brill-Noether loci that can be explicitly described.*

The purpose of our result is to better understand the birational geometry of these Lagrangian fibrations, though the algorithms we have developed will have much broader applications to other moduli spaces.

# 5 Other topics

## 5.1 Generalized complex geometry

Hitchin's generalized complex geometry [13] combines complex and symplectic geometry. Both complex and symplectic structures give examples, and for hyperkähler manifolds one can deform continuously between these different types. First-order deformations of a complex manifold $X$ regarded as a *generalized complex manifold* are parametrized by the degree two Hochschild cohomology

$$HT^2(X) := H^0(X, \wedge^2 T) \oplus H^1(X, T) \oplus H^2(X, \mathcal{O}).$$

This space also parametrizes deformations of the category $\mathrm{Coh}(X)$ of coherent sheaves, and Toda [54] thereby proved that a Fourier-Mukai transform $D^b(X) \to D^b(Y)$ induces a correspondence between deformations of $\mathrm{Coh}(X)$ and $\mathrm{Coh}(Y)$. Moreover, the Fourier-Mukai transform itself can be deformed. I connected Toda's work to generalized complex geometry.

**Theorem 5.1 ([40])** *There exist $\mathbb{P}^1$-families of generalized K3 surfaces, $X_t$ and $Y_t$, such that*

1. *$X_t$ are all complex K3 surfaces, whereas $Y_t$ are symplectic K3 surfaces with B-fields for $t \neq 0$,*
2. *$X_0$ and $Y_0$ are Fourier-Mukai dual elliptic K3 surfaces,*
3. *for $t \neq 0$, $X_t$ and $Y_t$ are mirror Strominger-Yau-Zaslow fibrations.*

The expectation is that these families will evince an example of a deformation of a Fourier-Mukai pair to a mirror pair (in the sense of Homological Mirror Symmetry) of complex and symplectic K3 surfaces. The derived equivalence must be extended beyond first-order deformations to complete this picture.

A hyperkähler manifold $M$ admits an $S^2$-family of compatible complex structures that can be combined into a single complex structure on $M \times S^2$, known as the *twistor space*. With my student Rebecca Glover, I extended this to the $S^2 \times S^2$-family of compatible generalized complex structures.

**Theorem 5.2 ([9])** *Given a hyperkähler manifold $M$, there exists a pair of generalized complex structures on $M \times S^2 \times S^2$ making it into a generalized pseudo-Kähler manifold.*

This *generalized twistor space* is an example of a generalized complex manifold whose type jumps from complex to symplectic at different points. My ongoing work aims to describe generalized holomorphic bundles on this space, to better understand deformations of $\mathrm{Coh}(X)$ induced by deformations of $X$ as a generalized complex manifold.

## 5.2 Recent work

Instead of Fourier-Mukai dual K3 surfaces $X$ and $Y$, one can consider a K3 surface $X$ and higher-dimensional moduli space $Y$ of sheaves on $X$. A universal sheaf will induce an *inclusion* $HT^2(X) \to HT^2(Y)$ and there can be gerby/non-commutative deformations of $X$, in $H^0(X, \wedge^2 T) \oplus H^2(X, \mathcal{O})$, that correspond to deformations of $Y$ as a usual complex manifold, in $H^1(Y, T)$. For example, for $Y = \mathrm{Hilb}^n X$ we have

$$\dim H^1(Y, T) = 21 > \dim H^1(X, T) = 20.$$

The additional deformation of $\mathrm{Hilb}^n X$ corresponds to a gerby/non-commutative deformation of $X$. I am currently trying to make these ideas rigorous by developing a Hilbert scheme construction for generalized complex manifolds. This would generalize work of Nekrasov-Schwarz and Nevins-Stafford, who constructed Hilbert schemes of non-commutative $\mathbb{C}^2$ and $\mathbb{P}^2$, respectively.

Twistor spaces exist for hyperkähler, hypercomplex, quaternion-Kähler, and quaternionic manifolds. I am extending my construction of generalized twistor spaces for hyperkähler manifolds to these other kinds of manifolds. For a quaternion-Kähler manifold $M$ my approach is to use Swann's construction, which associates to $M$ a hyperkähler manifold of dimension four more. The generalized twistor space of this hyperkähler manifold should admit an action of $\mathbb{H}^*$, whose quotient gives a generalized twistor space for $M$. Surprisingly, it seems this approach has not been elucidated even for usual twistor spaces.

### 5.3 Arithmetic applications

For a non-fine moduli space $S$ of stable sheaves on a variety $X$ there is an obstruction $\beta \in \mathrm{H}^2(S, \mathcal{O}^*)$ to the existence of a universal sheaf. This holomorphic gerbe $\beta$ has arithmetic applications, as it may also give a *Brauer-Manin obstructions* to the existence of rational points on $S$ over a number field $k$. Together with McKinnie, Tanimoto, and Várilly-Alvarado, I constructed examples of Fourier-Mukai dual K3 surfaces $X$ and $S$ for which $\beta$ can be realized as a Brauer-Severi $\mathbb{P}^n$-bundle over $S$ [19]. This geometric realization of $\beta$ allowed us to prove that it does indeed obstruct the existence of rational points on $S$.

## 6 New projects

### 6.1 Topological bounds on hyperkähler manifolds

Bounding the second Betti number represents an important step toward classifying hyperkähler manifolds, as $\mathrm{H}^2(X, \mathbb{Z})$ controls much of the geometry; see also the finiteness theorems of Huybrechts [14]. Guan proved that $b_2 \leq 23$ in dimension four, and I found the first bounds in dimensions six and eight.

**Theorem 6.1 ([49])** *Let $X$ be a compact hyperkähler manifold of (complex) dimension six or eight. Under certain hypotheses on the Looijenga-Lunts-Verbitsky (LLV) decomposition of the rational cohomology, the second Betti number is bounded above by* 23 *or* 24*, respectively.*

The proof uses representation theory, decomposing the cohomology into irreducible $\mathfrak{so}(4, b_2-2)$-modules. Kim-Laza extended this result to all dimensions, again under a hypothesis on the LLV decomposition.

A different approach uses Rozansky-Witten invariants, described earlier. Recently I found evidence that certain Rozansky-Witten invariants are always positive [48]. This evidence includes:

1. computations for all known hyperkähler manifolds up to and including dimension eight,
2. an *experimental proof* based on Maple computations with randomly generated curvature tensors.

This opens up a totally new way of studying hyperkähler manifolds: there are few known examples, but we can experiment with curvature tensors with the same symmetries as a hyperkähler manifold, and thereby discover new properties. A consequence of the conjectural Rozansky-Witten invariant bounds in [48] is an outright bound on the second Betti number in arbitrary dimension. One of my current goals is to establish the conjecture rigorously, not just 'experimentally'.

### 6.2 Eynard-Orantin topological recursion and integrable systems

Recently I started a new collaboration with Olivia Dumitrescu, Motohico Mulase, and Laura Schaposnik with the goal of unifying the study of Lagrangian fibrations, Hitchin systems, and Eynard-Orantin invariants (which is being funded by an NSF FRG grant). The latter provide a recursive method of calculating invariants associated to a curve. They were applied to the spectral curves of the GL-Hitchin system $M \to B$ by Dumitrescu-Mulase [8]. Remarkably, Baraglia-Huang proved that the special Kähler metric on the base $B$ can be computed from the Eynard-Orantin invariants of the spectral curves, and Norbury et al. used topological recursion to compute the Donagi-Markman cubic form on $B$, which describes the variation of the fibres. All of this suggests a deep relation between the individual spectral curves and the entire integrable system, and one of our goals is to extend this to Lagrangian fibrations by Prym varieties, described above.

For example, the key step to my classification results for Lagrangian fibrations by Jacobians [42, 43] is showing that the spectral curves lie in a K3 surface. We anticipate that the spectral curves of a Lagrangian fibration by Jacobians or Prym varieties will always lie in a surface, and topological recursion suggests a general method for recovering this surface. Note that if a curve of genus $g \geq 12$ lies in a K3 surface, then generically it will lie on a unique K3 surface; reconstructing this surface from the curve is a difficult problem. For a K3 curve, I suspect that the Eynard-Orantin topological recursion computes certain periods that are used to construct action-angle variables on the corresponding Beauville-Mukai system. In particular, this suggests that all of the 'information' of the K3 surface is already encoded in the curve, via the topological recursion invariants, and would provide a solution to the classical problem of recovering the surface from the curve.

## References


[1] A. Bayer and E. Macrì, *Projectivity and birational geometry of Bridgeland moduli spaces*, J. Amer. Math. Soc. **27** (2014), no. 3, 707–752.

[2] A. Beauville, *Counting rational curves on K3 surfaces*, Duke Math. J. **97** (1999), no. 1, 99–108.

[3] A. Beauville, *Vector bundles on Fano threefolds and K3 surfaces*, Boll. Unione Mat. Ital. (2020).

[4] A. Căldăraru, *Derived categories of twisted sheaves on Calabi-Yau manifolds*, PhD thesis, Cornell University, May 2000 (available from **www.math.upenn.edu/∼andreic/**).

[5] O. Debarre, *On the Euler characteristic of generalized Kummer varieties*, Amer. J. Math. **121** (1999), no. 3, 577–586.

[6] M. de Cataldo, T. Hausel, and L. Migliorini, *Topology of Hitchin systems and Hodge theory of character varieties: the case $A_1$*, Ann. of Math. (2) 175 (2012), no. 3, 1329–1407.

[7] R. Donagi, L. Ein, and R. Lazarsfeld, *Nilpotent cones and sheaves on K3 surfaces*, in Birational Algebraic Geometry (Baltimore 1996), Contemp. Math. **207**, Amer. Math. Soc., 1997, 51–61.

[8] O. Dumitrescu and M. Mulase, *Quantum curves for Hitchin fibrations and the Eynard-Orantin theory*, Lett. Math. Phys. **104** (2014), pp. 635–671.

[9] R. Glover and J. Sawon, *Generalized twistor spaces for hyperkähler manifolds*, J. London Math. Soc. (2) **91** (2015), no. 2, 321–342.

[10] T. Hausel and M. Thaddeus, *Mirror symmetry, Langlands duality, and the Hitchin system*, Invent. Math. **153** (2003), no. 1, 197–229.

[11] N. Hitchin, *Stable bundles and integrable systems*, Duke Math. J. **54** (1987), no. 1, 91–114.

[12] N. Hitchin and J. Sawon, *Curvature and characteristic numbers of hyperähler manifolds*, Duke Math. Jour. **106** (2001), no. 3, 599–615.

[13] N. Hitchin, *Generalized Calabi-Yau manifolds*, Q. J. Math. **54** (2003), no. 3, 281–308.

[14] D. Huybrechts, *Finiteness results for hyperkähler manifolds*, J. reine angew. Math. **558** (2003), 15–22.

[15] J.–M. Hwang and K. Oguiso, *Characteristic foliation on the discriminantal hypersurface of a holomorphic Lagrangian fibration*, Amer. J. Math. **131** (2009), 981–1007.

[16] M. Kontsevich, *Homological algebra of mirror symmetry*, in Proceedings of the International Congress of Mathematicians, Vol. **1**, **2** (Zürich, 1994), 120–139, Birkhäuser, Basel, 1995.

[17] P. Kruse, *Moduli spaces of Bridgeland stable objects on K3 surfaces and their $\mathbb{P}^3$ relatives*, PhD thesis, UNC-Chapel Hill, July 2020.

[18] D. Markushevich and A. Tikhomirov, *New symplectic V-manifolds of dimension four via the relative compactified Prymian*, Int. J. Math. **18** (2007), no. 10, 1187–1224.

[19] K. McKinnie, J. Sawon, S. Tanimoto, and A. Várilly-Alvarado, *Brauer groups on K3 surfaces and arithmetic applications*, in Brauer groups and obstruction problems, 177–218, Prog. in Math. **320**, Birkhäuser/Springer, Cham, 2017.

[20] S. Mukai, *Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves,* Nagoya Math. J. **81** (1981), 153–175.

[21] S. Mukai, *Symplectic structure of the moduli space of simple sheaves on an abelian or K3 surface*, Invent. Math. **77** (1984), 101–116.

[22] K. O'Grady, *Desingularized moduli spaces of sheaves on a K3*, J. reine angew. Math. **512** (1999), 49–117.

[23] X. Qin, *Compactification of the moduli space of minimal instantons on the Fano 3-fold $V_5$*, J. of Pure and Applied Alg. (2021), 22pp. DOI: 10.1016/j.jpaa.2020.106526

[24] X. Qin, *Compactification of the moduli space of minimal instantons on the Fano threefold $V_4$*, Eur. J. Math. (2021), 22 pp. DOI: 10.1007/s40879-021-00486-5

[25] X. Qin, *Bridgeland stability of minimal instantons on the Fano threefolds*, preprint **arXiv:2105.14617**.

[26] X. Qin and J. Sawon, *Birational geometry of Beauville-Mukai systems I: the rank three and genus two case*, preprint **arXiv:2207.12603**, 33pp.

[27] X. Qin and J. Sawon, *Birational geometry of Beauville-Mukai systems II: general theory in low ranks*, preprint **arXiv:2207.12608**, 28pp.

[28] X. Qin and J. Sawon, *Birational geometry of Beauville-Mukai systems III: asymptotic behavior*, preprint **arXiv:2210.03095**, 9pp.

[29] X. Qin and J. Sawon, *Toward a classification of $(1, 2)$-polarized Lagrangian fibrations*, draft, 9pp.

[30] J. Roberts and J. Sawon, *Generalizations of Rozansky-Witten invariants*, in Invariants of knots and 3-manifolds, Kyoto 2001, Geometry & Topology Monographs Vol. 4 (2002), 263–279.

[31] L. Rozansky and E. Witten, *Hyperkähler geometry and invariants of three-manifolds*, Selecta Math. **3** (1997), 401–458.

[32] J. Sawon, *The Rozansky-Witten invariants of hyperkähler manifolds*, Proc. 7th International Conference on Differential Geometry and Applications, Brno (1999), 429–436.

[33] J. Sawon, *A new weight system on chord diagrams via hyperkähler geometry*, in Quaternionic Structures in Mathematics and Physics, Rome, September 1999, World Scientific (2001), 349–363.

[34] J. Sawon, *Topological quantum field theory and hyperkähler geometry*, Turkish J. Math. **25** (2001), no. 1, 169–194. (Proc. of the 7th Gökova Geometry-Topology Conference, June 2000.)

[35] J. Sawon, *Abelian fibred holomorphic symplectic manifolds*, Turkish Jour. Math. **27** (2003), no. 1, 197–230. (Proc. of the 9th Gökova Geometry-Topology Conference, May 2002.)

[36] J. Sawon, *Derived equivalence of holomorphic symplectic manifolds*, in Algebraic structures and moduli spaces (Montreal, July 2003), CRM Proc. & Lecture Notes Series **38** (2004), 193–211.

[37] J. Sawon, *Lagrangian fibrations on Hilbert schemes of points on K3 surfaces*, J. Algebraic Geom. **16** (2007), no. 3, 477–497.

[38] J. Sawon, *On the discriminant locus of a Lagrangian fibration*, Math. Ann. **341** (2008), no. 1, 201–221.

[39] J. Sawon, *Twisted Fourier-Mukai transforms for holomorphic symplectic four-folds*, Adv. Math. **218** (2008), no. 3, 828–864.

[40] J. Sawon, *Fourier-Mukai transforms, mirror symmetry, and generalized K3 surfaces*, preprint **arXiv:1209.3202**, 26pp.

[41] J. Sawon, *Isotrivial elliptic K3 surfaces and Lagrangian fibrations*, preprint **arXiv:1406.1233**, 17pp.

[42] J. Sawon, *On Lagrangian fibrations by Jacobians I*, J. reine angew. Math. **701** (2015), 127–151.

[43] J. Sawon, *On Lagrangian fibrations by Jacobians II*, Commun. Contemp. Math. **17** no. 5 (2015), 1450046, 23 pp.

[44] J. Sawon, *A finiteness theorem for Lagrangian fibrations*, J. Alg. Geom. **25** (2016), no. 3, 431–459.

[45] J. Sawon, *Moduli spaces of sheaves on K3 surfaces*, J. Geom. and Phys. **109** (2016), 68–82.

[46] J. Sawon, *Lagrangian fibrations by Prym varieties*, Matemática Contemporânea **47** (2020), 182–227.

[47] J. Sawon, *Singular fibres of generic Lagrangian fibrations*, Commun. Contemp. Math. (2021), 19 pp. DOI: 10.1142/S021919972150070X

[48] J. Sawon, *Topological bounds on hyperkähler manifolds*, preprint **arXiv:2112.11617**, 22pp.

[49] J. Sawon, *A bound on the second Betti number of hyperkähler manifolds of complex dimension six*, Eur. J. Math. **8** (2022), 1196–1212. DOI: 10.1007/s40879-021-00526-0

[50] J. Sawon and C. Shen, *Deformations of compact Prym fibrations to Hitchin systems*, Bull. Lond. Math. Soc. **54** (2022), no. 5, 1568–1583. DOI: 10.1112/blms.12643

[51] J. Sawon and C. Shen, *A singular Lagrangian fibration by Prym varieties, I. construction*, first draft, 15pp.

[52] C. Shen, *Lagrangian fibrations by Prym varieties*, PhD thesis, UNC-Chapel Hill, March 2020.

[53] A. Strominger, S-T. Yau, E. Zaslow, *Mirror symmetry is T-duality*, Nuclear Phys. **B 479** (1996), 243–259.

[54] Y. Toda, *Deformations and Fourier-Mukai transforms*, J. Differential Geom. **81** (2009), no. 1, 197–224.