Lagrangian fibrations Part II : six dimensions

Justin Sawon[1]

THE UNIVERSITY of NORTH CAROLINA at CHAPEL HILL

Seminar Algebraic Geometry
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Overview

- generalities on Lagrangian fibrations
- smooth holomorphic symplectic sixfolds
- polarization types
- fibrations by Prym varieties
- dual fibrations

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Holomorphic symplectic manifolds

Let X be a compact Kähler manifold with $c_1 = 0$.

Thm (Bogomolov): $\exists$ finite étale cover $\tilde{X}$ of X with

$$\tilde{X} = T \times \prod_i CY_i \times \prod_j IHS_j,$$

$T =$ torus, $CY_i =$ (strict) Calabi-Yau manifolds, and $IHS_j = \ldots$

Def: A compact Kähler manifold X is a *holomorphic symplectic manifold* if it admits a non-degenerate holomorphic two-form σ. In addition if $\pi_1(X) = 0$ and $\mathrm{H}^0(\Omega^2)$ is generated by σ then we say X is an *irreducible holomorphic symplectic (IHS) manifold*.

Examples of IHS manifolds

1. Hilbert schemes of points on K3 surfaces, $\mathrm{Hilb}^n S \to \mathrm{Sym}^n S$.

2. Generalized Kummer varieties, $\widetilde{\mathrm{Hilb}^{n+1}} A = A \times K_n(A)$.
Equivalently $K_n(A) :=$ kernel$(\mathrm{Hilb}^{n+1} A \longrightarrow \mathrm{Sym}^{n+1} A \longrightarrow A)$.

3. Mukai moduli spaces of stable sheaves on K3/abelian surfaces.

$$\mathrm{Ext}^1(\mathcal{E},\mathcal{E}) \times \mathrm{Ext}^1(\mathcal{E},\mathcal{E}) \to \mathrm{Ext}^2(\mathcal{E},\mathcal{E}) \xrightarrow{\mathrm{tr}} \mathrm{H}^2(\mathcal{O}) \cong \mathbb{C}$$

4. O'Grady's spaces, OG6 and OG10.

Three examples known in dimension six: $\mathrm{Hilb}^3 S$, $K_3(A)$, and OG6.

Fibrations

Let X be an IHS manifold of dimension $2n$.

Thm (Matsushita): If $X \to B$ is a proper fibration then

1. $\dim B = n = \dim F$,
2. F is Lagrangian wrt the holomorphic symplectic form σ,
3. generic fibre is a complex torus.

Rmk: Lagrangian means $TF \subset TX$ is maximal isotropic wrt σ. Integrable means $T^*B \subset T^*X$ is maximal isotropic wrt σ^{-1}. Thus Lagrangian fibrations are equivalent to integrable systems.

Thm (Hwang): B is isomorphic to $\mathbb{P}^n$ if it is smooth.

Rmk: Hodge theory $\implies$ general fibre is an abelian variety.

Polarizations of abelian varieties

A polarization H of an abelian variety gives

$$c_1(H) \in \mathrm{H}^2(A, \mathbb{Z}) = \Lambda^2 \mathrm{H}_1(A, \mathbb{Z})^*.$$

With respect to a standard basis

$$c_1(H) = \begin{bmatrix} & & & & d_1 & & & \\ & 0 & & & & d_2 & & \\ & & & & & & \ddots & \\ & & & & & & & d_n \\ -d_1 & & & & & & & \\ & -d_2 & & & & & 0 & \\ & & \ddots & & & & & \\ & & & -d_n & & & & \end{bmatrix}$$

with $d_1 | d_2 | \cdots | d_n$. We call this the *type* of the polarization.

Examples for $\mathrm{Hilb}^n S$

1. If $S \longrightarrow \mathbb{P}^1$ is an elliptic K3 surface then the Hilbert scheme

$$\mathrm{Hilb}^n S \to \mathrm{Sym}^n S \to \mathrm{Sym}^n \mathbb{P}^1 = \mathbb{P}^n$$

is a Lagrangian fibration. Its fibres look like

$$E_1 \times E_2 \times \cdots \times E_n.$$

2. Beauville-Mukai system: Let C be a genus n curve in a K3 surface S, with $|C| \cong \mathbb{P}^n$ and $\mathcal{C}/\mathbb{P}^n$ the family of curves linearly equivalent to C.

$$X := \overline{\mathrm{Jac}}^d(\mathcal{C}/\mathbb{P}^n) \longrightarrow \mathbb{P}^n$$

is a Lagrangian fibration, deformation equivalent to $\mathrm{Hilb}^n S$.

Or $X \cong$ moduli space $M(0, [C], 1 - g + d)$ of stable sheaves on S.

Examples for $K_n(A)$

3. Isotrivial system: If $A = E \times F$ then

$$\mathrm{Hilb}^{n+1} A \longrightarrow \mathrm{Sym}^{n+1} A \longrightarrow \mathrm{Sym}^{n+1} E \cong J^{n+1} E \times \mathbb{P}^n \cong E \times \mathbb{P}^n$$

induces a Lagrangian fibration $K_n(A) \longrightarrow \mathbb{P}^n$ with smooth fibres

$$\cong \{(f_0, f_1, \ldots, f_n) \in F^{n+1} \mid f_0 + f_1 + \ldots + f_n = 0 \text{ in } F\}.$$

Rmk: This fibre is complementary to

$$\Delta : F \cong \{(f, f, \ldots, f) \mid f \in F\} \longrightarrow F^{n+1}$$

and therefore has polarization type $(1, 1, \ldots, 1, n+1)$.

Examples for $K_n(A)$

4. Debarre system: Let $C \subset A$ give a polarization of type $(1, n+1)$. Then C has genus $n+2$ and $|C| \cong \mathbb{P}^n$. Let $\mathcal{C}/\mathbb{P}^n$ be the family of curves linearly equivalent to C and

$$Y := \overline{\mathrm{Jac}}^d(\mathcal{C}/\mathbb{P}^n) \longrightarrow \mathbb{P}^n.$$

Or $Y \cong$ moduli space $M(0, [C], d-n-1)$ of stable sheaves on A. Then

$$X := \mathrm{kernel}(\mathrm{Alb} : Y \longrightarrow A)$$

is a Lagrangian fibration with fibres

$$X_t = \mathrm{kernel}(\overline{\mathrm{Jac}}^d C_t \longrightarrow A).$$

Rmk: Polarization type of the fibres is $(1, 1, \ldots, 1, n+1)$.

Examples for $K_n(A)$ and OG6

5. Debarre system/OG6: Let $C \subset A$ give a polarization of type $(2,2)$. Then C has genus 5 and $|C| \cong \mathbb{P}^3$. Consider

$$Y := \overline{\mathrm{Jac}}^d(\mathcal{C}/\mathbb{P}^3) \longrightarrow \mathbb{P}^3,$$

i.e., $Y \cong M(0,[C],d-4)$ on A, and $X := \text{kernel}(\mathrm{Alb} : Y \longrightarrow A)$.

- If d is odd then X is deformation equivalent to $K_3(A)$,
- If d is even then $\widetilde{X}$ is deformation equivalent to OG6.

Rmk: Both cases have fibres of polarization type $(1,2,2)$.

Summary of six-dimensional examples

Example	**Polarization type**		
Hilb^3 of elliptic K3	$(1, 1, 1)$		
Beauville-Mukai system	$(1, 1, 1)$		
Isotrivial system on $K_3(A)$	$(1, 1, 4)$		
Debarre system for $A^{1,4}$	$(1, 1, 4)$		
Debarre system for $A^{2,2}$	$(1, 2, 2)$		
O'Grady 6 on $A^{2,2}$	$(1, 2, 2)$		

An invariant of Lagrangian fibrations

Thm (Wieneck): In a family of Lagrangian fibrations the polarization type of the fibres is constant.

For fibrations on $\mathrm{Hilb}^n K3$ the polarization is *principal*

$$(d_1, d_2, \ldots, d_n) = (1, 1, \ldots, 1).$$

For fibrations on $K_n(A)$ the polarization is of type

$$(d_1, \ldots, d_{n-1}, d_n) = (1, \ldots, 1, d_{n-1}, d_n)$$

with $d_{n-1}d_n = n + 1$.

Thm (Markman): Every Lagrangian fibration on $\mathrm{Hilb}^n K3$ is a Beauville-Mukai system, up to a *Tate-Shafarevich twist*.

Fibrations by Jacobians

Thm (S-): Let $\mathcal{C}/\mathbb{P}^3$ be a flat family of reduced curves of genus 3 such that $X = \overline{\mathrm{Jac}}^d(\mathcal{C}/\mathbb{P}^3)$ is a Lagrangian fibration. If the curves

- are irreducible Gorenstein non-hyperelliptic, or
- are canonically positive 2-connected hyperelliptic,

then $X/\mathbb{P}^3$ must be a Beauville-Mukai integrable system.

Rmk: The general principally polarized abelian threefold is the Jacobian of a (non-hyperelliptic) curve of genus three.

Qu: If the general fibre of $X/\mathbb{P}^3$ is the product $E_1 \times E_2 \times E_3$ of elliptic curves, must X be Hilb^3 of an elliptic K3?

Fibrations by Jacobians

Proof (in the non-hyperelliptic case): The relative canonical embedding gives

$$\mathcal{C} \to \mathbb{P}(R^1\pi_*\mathcal{O}_{\mathcal{C}}) = \mathbb{P}(R^1\pi_*\mathcal{O}_X) = \mathbb{P}(\Omega^1_{\mathbb{P}^3}) \subset \mathbb{P}^3 \times (\mathbb{P}^3)^\vee.$$

Indeed $\mathcal{C}$ is the zero locus in $\mathbb{P}(\Omega^1_{\mathbb{P}^3})$ of a section of

$$\mathcal{O}_{\mathbb{P}(\Omega^1)}(4) \otimes \pi^*\mathcal{O}_{\mathbb{P}^3}(k) = \mathcal{O}(k+4,4)|_{\mathbb{P}(\Omega^1)}.$$

Now $R^1\pi_*\mathcal{O}_{\mathcal{C}} = \Omega^1_{\mathbb{P}^3}$ determines $k = -4$, so

$$\mathcal{O}(k+4,4)|_{\mathbb{P}(\Omega^1)} = \mathcal{O}(0,4)|_{\mathbb{P}(\Omega^1)}$$

is pulled back from $(\mathbb{P}^3)^\vee$. This means the curves are hyperplane sections of a quartic K3 surface in $(\mathbb{P}^3)^\vee$.

Finiteness

Thm (S-): Fixing $d_1 | \dots | d_n$, there are finitely many Lagrangian fibrations up to deformation with

- polarization type $(d_1, \dots, d_n)$,
- a global section,
- maximally varying fibres,
- semistable singular fibres in codimension one.

Thm (S-): In each dimension $2n$ the overall degree $d_1 \cdots d_n$ of the fibres can take only finitely many values up to an n^{th} power k^n.

(Generalized) Prym varieties

Let $\pi : C \to D$ be a double cover of curves with covering involution τ. Then

$$\mathrm{Fix}^0(\tau^*) = \pi^*\mathrm{Jac}^0 D \subset \mathrm{Jac}^0 C.$$

Def: The Prym variety of C/D is

$$\mathrm{Prym}(C/D) := \mathrm{Fix}^0(-\tau^*),$$

an abelian variety of dimension $g_C - g_D$ and polarization type

$$(\underbrace{1,\ldots,1}_{g_C - 2g_D}, \underbrace{2,\ldots,2}_{g_D}).$$

$\mathrm{Prym}(C/D)$ is principally polarized iff $\pi : C \to D$ has zero or two branch points.

Families of Prym varieties

Let $\pi : S \to T$ be a K3 double cover of another surface with *anti-symplectic* covering involution τ. A curve $D \subset T$ has a double cover $C \subset S$,

$$\begin{array}{ccc} C & \subset & S \\ {\scriptstyle 2:1}\downarrow & & {\scriptstyle 2:1}\downarrow \\ D & \subset & T. \end{array}$$

Let $\mathcal{D} \to |D|$ be the complete linear system in T, $\mathcal{C} = \pi^*\mathcal{D}$.

Thm (Markushevich-Tikhomirov, Arbarello-Saccà-Ferretti, Matteini): We can construct a relative Prym variety

$$\mathrm{Prym}(\mathcal{C}/\mathcal{D}) := \mathrm{Fix}^0(\mathcal{E} \mapsto \mathcal{E}xt^1_S(\tau^*\mathcal{E}, \mathcal{O}(-C))) \subset \overline{\mathrm{Jac}}^0(\widetilde{\mathcal{C}}/|C|).$$

This is a symplectic variety and a Lagrangian fibration over $|D|$.

Examples

1. Markushevich-Tikhomirov system: S/T a K3 double cover of a degree two del Pezzo, C/D a genus three cover of an elliptic curve, $\mathrm{Prym}(C/D)$ an abelian surface of type $(1,2)$.

Then $\mathrm{Prym}(\mathcal{C}/\mathcal{D}) \to \mathbb{P}^2$ is an *irreducible* symplectic orbifold of $\dim^{\mathrm{n}}$ four, with 28 isolated singularities that look like $\mathbb{C}^4/\pm 1$.

2. Arbarello-Saccà-Ferretti system: S/T a K3 double cover of an Enriques, D genus $n+1$, $\mathrm{Prym}(C/D)$ principally polarized.

Then $\mathrm{Prym}(\mathcal{C}/\mathcal{D}) \to \mathbb{P}^n$ is a symplectic variety, which is

- birational to $\mathrm{Hilb}^n K3$ if D is hyperelliptic,
- simply connected, no symplectic resolution, otherwise,
- and irreducible if n is even.

Qu: Why does the classification for Jacobian fibrations not apply?

Examples

3. Matteini system: S/T a K3 double cover of a cubic del Pezzo, C/D a genus four cover of an elliptic curve, $\mathrm{Prym}(C/D)$ an abelian threefold of type $(1,1,2)$.

$\mathrm{Prym}(\mathcal{C}/\mathcal{D}) \to \mathbb{P}^3$ is an *irreducible* symplectic orbifold of $\dim^{\mathrm{n}}$ six, with singularities that look like $\mathbb{C}^2 \times (\mathbb{C}^4/\pm 1)$ and $\mathbb{C}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$.

Questions:

- What are the structure of the singularities?
- Are these varieties simply connected? Are they irreducible?

A six-dimensional example

S/T a K3 double cover of a degree one del Pezzo, $D \in |-2K_T|$, C/D a genus five cover of a genus two curve. Then

$$\mathrm{Prym}(\mathcal{C}/\mathcal{D}) := \mathrm{Fix}^0(-) \subset \overline{\mathrm{Jac}}^0(\widetilde{\mathcal{C}}/|C|) \leftarrow \mathrm{OG10}$$

is a symplectic variety of $\dim^{\mathrm{n}}$ six and a Lagrangian fibration with abelian fibres of type $(1,2,2)$ over $|D| \cong \mathbb{P}^3$.

Lemma (Arbarello et al.): If $C = C_1 \cup C_2$ with $C_1.C_2 = 2k$ then $\mathrm{Prym}(\mathcal{C}/\mathcal{D})$ looks locally like $\mathbb{C}^{N-2k} \times (\mathbb{C}^{2k}/\pm 1)$ at $[\mathcal{F}_1 \oplus \mathcal{F}_2]$.

Thm (S-Shen): $\mathrm{Prym}(\mathcal{C}/\mathcal{D})$ contains 120 isolated singularities that look like $\mathbb{C}^6/\pm 1$ (and thus there is no symplectic resolution).

A birational model

The del Pezzo T is a double cover of the quadric cone Q. The covering involution lifts to another anti-symplectic involution on S:

$$
\begin{array}{ccccccc}
 & & S & & & & \widetilde{S} = \text{resolution of } \overline{S} \\
 & \swarrow & \downarrow & \searrow & & \swarrow & \\
T & & \mathbb{P}^2 & & \overline{S} & & \\
 & \searrow & \downarrow & \swarrow & & & \\
 & & Q & & & &
\end{array}
$$

The anti-symplectic involutions commute and their composition gives a symplectic involution on S, with quotient a singular K3 surface $\overline{S}$ with 8 A_1-singularities.

A birational model

$$\begin{array}{ccccccccc} & & & C\subset S & & & & & \tilde{C}\subset\tilde{S} \\ & & \swarrow & \downarrow & \searrow & & & \swarrow & \\ D\subset T & & & \mathbb{P}^2 & & & \overline{C}\subset\overline{S} & & \\ & & \searrow & \downarrow & \swarrow & & & & \\ & & & Q & & & & & \end{array}$$

A generic τ-invariant $C \subset S$ is an étale double cover of a genus three curve $\overline{C} \subset \overline{S}$, which is isomorphic to $\tilde{C} \subset \tilde{S}$.

Pull-back induces a map

$$\mathrm{Jac}^0\tilde{C} = \mathrm{Jac}^0\overline{C} \longrightarrow \mathrm{Jac}^0 C$$

which is two-to-one onto its image $\mathrm{Prym}(C/D)$.

A birational model

Let $\widetilde{\mathcal{M}} := \overline{\mathrm{Jac}}^0(\tilde{\mathcal{C}}/\mathbb{P}^3)$ be the Beauville-Mukai system of $\tilde{C} \subset \tilde{S}$. Then there is a rational dominant generically two-to-one map

$$\widetilde{\mathcal{M}} \dashrightarrow \mathrm{Prym}(\mathcal{C}/\mathcal{D}).$$

Moreover, $\widetilde{\mathcal{M}}$ is deformation equivalent to $\mathrm{Hilb}^3 \tilde{S}$.

Thm (S-Shen): For $\mathrm{Prym}(\mathcal{C}/\mathcal{D})$ we have

- the symplectic structure is unique up to a scalar, $h^{2,0} = 1$,
- vanishing $h^{1,0} = 0$.

Rmk: We say that $\mathrm{Prym}(\mathcal{C}/\mathcal{D})$ is a *primitive* symplectic variety.

Summary of six-dimensional examples

Example	**Polarization type**		
Hilb^3 of elliptic K3	$(1,1,1)$		
Beauville-Mukai system	$(1,1,1)$		
ASF system (non-hyperelliptic)	$(1,1,1)$		
Matteini system	$(1,1,2)$		
Isotrivial system on $K_3(A)$	$(1,1,4)$		
Debarre system for $A^{1,4}$	$(1,1,4)$		
Debarre system for $A^{2,2}$	$(1,2,2)$		
O'Grady 6 on $A^{2,2}$	$(1,2,2)$		
S-Shen system	$(1,2,2)$		

Dual fibrations for principal polarizations

The dual $\hat{A}$ of A has polarization type

$$\left(\frac{d_1 d_n}{d_n}, \frac{d_1 d_n}{d_{n-1}}, \ldots, \frac{d_1 d_n}{d_2}, \frac{d_1 d_n}{d_1}\right).$$

Principally polarized abelian varieties are self-dual: for smooth C

$$\widehat{\mathrm{Jac}^0 C} := \mathrm{Pic}^0(\mathrm{Jac}^0 C) \cong \mathrm{Jac}^0 C.$$

Thm (Esteves-Kleiman): For integral C with surficial singularities

$$\overline{\mathrm{Pic}}^0(\overline{\mathrm{Jac}}^0 C) \cong \overline{\mathrm{Jac}}^0 C.$$

Cor: If $X \to \mathbb{P}^n$ is a Lagrangian fibration by Jacobians with a global section then $\hat{X} \cong X$ (at least over smooth and mildly singular fibres).

Dual fibrations of Debarre system

Recall the Debarre system constructed from $C \subset A^{1,n+1}$. Dualizing

$$0 \longrightarrow X_t \longrightarrow \overline{\mathrm{Jac}}^0 C \longrightarrow A \longrightarrow 0$$

gives

$$0 \longrightarrow \hat{A} \longrightarrow \widehat{\overline{\mathrm{Jac}}^0 C} \cong \overline{\mathrm{Jac}}^0 C \longrightarrow \hat{X}_t \longrightarrow 0.$$

Thm (S-): $\hat{A} = \mathrm{Pic}^0 A$ acts naturally on $Y := \overline{\mathrm{Jac}}^0(\mathcal{C}/\mathbb{P}^n)$ and $\hat{X} := [Y/\hat{A}]$ is an orbifold.

Thm (Kim):

$$\Gamma := \mathrm{kernel}(X_t \longrightarrow \hat{X}_t) = \mathrm{kernel}(\hat{A} \to A) \cong \mathbb{Z}/(n+1) \oplus \mathbb{Z}/(n+1)$$

acts fibrewise on $X/\mathbb{P}^n$ and $\hat{X} := [X/\Gamma]$.

Summary of six-dimensional examples

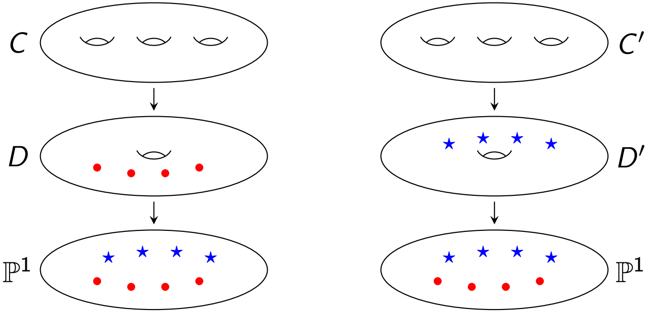
Example	**Type**		**Dual**
Hilb^3 of elliptic K3	$(1,1,1)$	$(1,1,1)$	Hilb^3 of elliptic K3
Beauville-Mukai	$(1,1,1)$	$(1,1,1)$	Beauville-Mukai
ASF system	$(1,1,1)$	$(1,1,1)$	ASF system
Matteini system	$(1,1,2)$		
Isotrivial on $K_3(A)$	$(1,1,4)$	$(1,4,4)$	Kim/S-
Debarre for $A^{1,4}$	$(1,1,4)$	$(1,4,4)$	Kim/S-
Debarre for $A^{2,2}$	$(1,2,2)$	$(1,1,2)$	Kim
O'Grady 6 on $A^{2,2}$	$(1,2,2)$	$(1,1,2)$	Kim
S-Shen system	$(1,2,2)$		

Pantazis's bigonal construction

Given a tower $C \xrightarrow{2:1} D \xrightarrow{2:1} \mathbb{P}^1$ we can construct $C' \xrightarrow{2:1} D' \xrightarrow{2:1} \mathbb{P}^1$

$$C' := \{\text{pairs of lifts } (c_1, c_3), (c_1, c_4), (c_2, c_3), (c_2, c_4)\}.$$

This interchanges the branch points of the double covers.



Thm (Pantazis): $\mathrm{Prym}(C'/D')$ is dual to $\mathrm{Prym}(C/D)$.

Dual of the Markushevich-Tikhomirov system

A K3 double cover S/T of a degree two del Pezzo is given by two quartics Δ and Δ' in $\mathbb{P}^2$ that are tangent at eight points.

- $f : T \to \mathbb{P}^2$ is a double cover branched over Δ
- $S \to T$ is branched over one component of $f^{-1}(\Delta')$

Applying the bigonal construction gives $S' \xrightarrow{2:1} T' \xrightarrow{2:1} \mathbb{P}^2$, with the roles of the quartics Δ and Δ' switched.

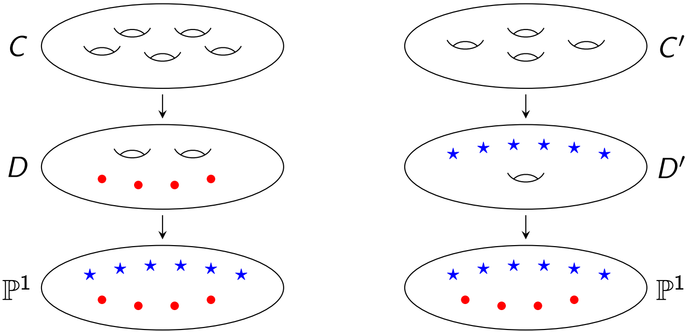
Thm (Menet): $\mathrm{Prym}(\mathcal{C}'/\mathcal{D}')$ over $\mathbb{P}^2$ is dual to $\mathrm{Prym}(\mathcal{C}/\mathcal{D})$.

Thus the dual of a Markushevich-Tikhomirov system is another Markushevich-Tikhomirov system.

Qu: Is there a relation between $D^b(S)$ and $D^b(S')$?

Dual of our fibration

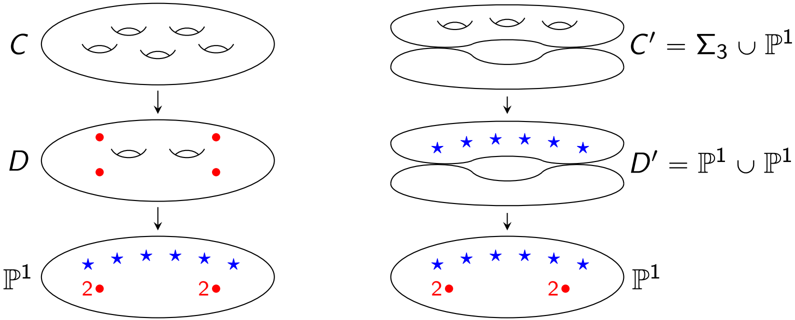
Fibres are $\mathrm{Prym}(C/D)$ with $g_C = 5$ and $g_D = 2$. Pantazis gives:



Thus $\mathrm{Prym}(C'/D')$ look like fibres of the Matteini system.

Questions: How to go from $S \xrightarrow{2:1} T \xrightarrow{2:1} Q$ to $S' \xrightarrow{2:1} T' \xrightarrow{???} Q$.

Dual of our fibration



The dual of the abelian threefold $\mathrm{Prym}(C/D)$ is

$$\begin{array}{ccccc} \mathrm{Prym}(C'/D') & \hookrightarrow & \mathrm{Jac}^0 C' & \subset & \overline{\mathrm{Jac}}^0 C' \\ & \overset{2:1}{\searrow} & \downarrow & & \\ & & \mathrm{Jac}^0 \Sigma_3. & & \end{array}$$

Dual of our fibration

Compare

$$\mathrm{Jac}^0 \tilde{C} = \mathrm{Jac}^0 \overline{C} \longrightarrow \mathrm{Jac}^0 C,$$

two-to-one onto $\mathrm{Prym}(C/D)$, to

$$\mathrm{Prym}(C'/D') \longrightarrow \mathrm{Jac}^0 \Sigma_3.$$

In fact, $\tilde{C} = \overline{C}$ and Σ_3 are the same curve!

Start with $S \xrightarrow{2:1} T \xrightarrow{2:1} Q$, a K3 double cover of a degree one del Pezzo cover of a quadric cone. The bigonal construction gives

$$S' = \overline{S} \cup \mathbb{P}^2 \xrightarrow{2:1} T' = Q \cup Q \xrightarrow{2:1} Q.$$

Rmk: Though S' and T' are not normal, and all curves C' and D' are reducible, $\mathrm{Prym}(C'/D')$ is smooth in general.

Dual of our fibration

Thm (S-): $\mathrm{Prym}(\mathcal{C}'/\mathcal{D}')$ over $\mathbb{P}^3$ is dual to $\mathrm{Prym}(\mathcal{C}/\mathcal{D})$.

Rmk: $\mathrm{Prym}(\mathcal{C}'/\mathcal{D}')$ is a double cover of the same Beauville-Mukai system that $\mathrm{Prym}(\mathcal{C}/\mathcal{D})$ is a $\mathbb{Z}/2\mathbb{Z}$ quotient of.

Question: Is S'/T' a degeneration of a K3 double cover of a cubic del Pezzo? Is $\mathrm{Prym}(\mathcal{C}'/\mathcal{D}')$ a degeneration of the Matteini system?

Parameters

The Matteini system depends on 13 parameters:

- 4 parameters for the cubic surface,
- 9 parameters for the branch locus in $|-2K|$.

Our system depends on 11 parameters:

- 8 parameters for the degree one del Pezzo T,
- 3 parameters for the branch locus in $|-2K_T|$.

However, $\mathrm{Prym}(\mathcal{C}/\mathcal{D})$ has two singular loci, locally $\mathbb{C}^4 \times (\mathbb{C}^2/\pm 1)$. Resolving creates two more divisors, and thus two more parameters.

Question: Is a general deformation of $\widetilde{\mathrm{Prym}(\mathcal{C}/\mathcal{D})}$ dual to the Matteini system?

Summary of six-dimensional examples

Example	Type		Dual
Hilb^3 of elliptic K3	$(1,1,1)$	$(1,1,1)$	Hilb^3 of elliptic K3
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ASF system	$(1,1,1)$	$(1,1,1)$	ASF system
Matteini system	$(1,1,2)$		
Isotrivial on $K_3(A)$	$(1,1,4)$	$(1,4,4)$	Kim/S-
Debarre for $A^{1,4}$	$(1,1,4)$	$(1,4,4)$	Kim/S-
Debarre for $A^{2,2}$	$(1,2,2)$	$(1,1,2)$	Kim
O'Grady 6 on $A^{2,2}$	$(1,2,2)$	$(1,1,2)$	Kim
S-Shen system	$(1,2,2)$	$(1,1,2)$	degenerate Matteini?