Lagrangian fibrations in four and six dimensions

Justin Sawon[1]

THE UNIVERSITY of NORTH CAROLINA at CHAPEL HILL

Algebraic Geometry Seminar
Mainz, 19 June, 2023

[1]Supported by MPIM Bonn and NSF awards DMS-1555206, DMS-2152130.

Overview

- generalities on Lagrangian fibrations
- examples in four dimensions
- examples in six dimensions
- classification results
- polarization types

(partly joint work with Chen Shen and Xuqiang Qin)

Holomorphic symplectic manifolds

Let X be a compact Kähler manifold with $c_1 = 0$.

Thm (Bogomolov): $\exists$ finite étale cover $\widetilde{X}$ of X with

$$\widetilde{X} = T \times \prod_i CY_i \times \prod_j IHS_j,$$

T = torus, CY_i = (strict) Calabi-Yau manifolds, and $IHS_j = \ldots$

Def: A compact Kähler manifold X is a *holomorphic symplectic manifold* if it admits a non-degenerate holomorphic two-form σ.

In addition if $\pi_1(X) = 0$ and $\mathrm{H}^0(\Omega^2)$ is generated by σ then we say X is an *irreducible holomorphic symplectic (IHS) manifold*.

Examples of IHS manifolds

1. Hilbert schemes of points on K3 surfaces, $\mathrm{Hilb}^n S \to \mathrm{Sym}^n S$.

2. Generalized Kummer varieties, $\widetilde{\mathrm{Hilb}^{n+1}} A = A \times K_n(A)$.
Equivalently $K_n(A) := \mathrm{kernel}(\mathrm{Hilb}^{n+1} A \longrightarrow \mathrm{Sym}^{n+1} A \longrightarrow A)$.

3. Mukai moduli spaces of stable sheaves on K3/abelian surfaces.

$$\mathrm{Ext}^1(\mathcal{E},\mathcal{E}) \times \mathrm{Ext}^1(\mathcal{E},\mathcal{E}) \to \mathrm{Ext}^2(\mathcal{E},\mathcal{E}) \xrightarrow{\mathrm{tr}} \mathrm{H}^2(\mathcal{O}) \cong \mathbb{C}$$

4. O'Grady's spaces, OG6 and OG10.

Up to deformation, two/three (smooth) examples known in dimensions four/six: $\mathrm{Hilb}^2 S$, $K_2(A)$, $\mathrm{Hilb}^3 S$, $K_3(A)$, and OG6.

Fibrations

Let X be an IHS manifold of dimension $2n$.

Thm (Matsushita): If $X \to B$ is a proper fibration then

1. $\dim B = n = \dim F$,
2. F is Lagrangian wrt the holomorphic symplectic form σ,
3. generic fibre is a complex torus.

Thm (Hwang): B is isomorphic to $\mathbb{P}^n$ if it is smooth.

Thm (Huybrechts-Xu): B is smooth if $n = 2$, thus $B \cong \mathbb{P}^2$.

Rmk (Voisin): Hodge theory $\implies$ general fibre is an abelian variety.

Polarizations of abelian varieties

A polarization H of an abelian variety gives

$$c_1(H) \in \mathrm{H}^2(A,\mathbb{Z}) = \Lambda^2 \mathrm{H}_1(A,\mathbb{Z})^*.$$

With respect to a standard basis

$$c_1(H) = \begin{bmatrix} & & & & d_1 & & & \\ & & 0 & & & d_2 & & \\ & & & & & & \ddots & \\ & & & & & & & d_n \\ -d_1 & & & & & & & \\ & -d_2 & & & & & 0 & \\ & & \ddots & & & & & \\ & & & -d_n & & & & \end{bmatrix}$$

with $d_1 | d_2 | \cdots | d_n$. We call this the *type* of the polarization.

Examples for $\mathrm{Hilb}^n S$

1a. Beauville-Mukai system: Let C be a genus n curve in a K3 surface S, with $|C| \cong \mathbb{P}^n$ and $\mathcal{C}/\mathbb{P}^n$ the family of curves linearly equivalent to C.

$$X := \overline{\mathrm{Jac}}^d(\mathcal{C}/\mathbb{P}^n) \longrightarrow \mathbb{P}^n$$

is a Lagrangian fibration, deformation equivalent to $\mathrm{Hilb}^n S$.

Or $X \cong$ moduli space $M(0, [C], 1-g+d)$ of stable sheaves on S.

1b. If $S \longrightarrow \mathbb{P}^1$ is an elliptic K3 surface then the Hilbert scheme

$$\mathrm{Hilb}^n S \to \mathrm{Sym}^n S \to \mathrm{Sym}^n \mathbb{P}^1 = \mathbb{P}^n$$

is a Lagrangian fibration. Its fibres look like

$$E_1 \times E_2 \times \cdots \times E_n.$$

(Generalized) Prym varieties

Let $\pi : C \to D$ be a double cover of curves with covering involution τ. Then

$$\mathrm{Fix}^0(\tau^*) = \pi^* \mathrm{Jac}^0 D \subset \mathrm{Jac}^0 C.$$

Def: The Prym variety of C/D is an abelian variety

$$\mathrm{Prym}(C/D) := \mathrm{Fix}^0(-\tau^*),$$

of dimension $g_C - g_D$, principally polarized if $\pi : C \to D$ has zero or two branch points, otherwise polarization type

$$(\underbrace{1, \ldots, 1}_{g_C - 2g_D}, \underbrace{2, \ldots, 2}_{g_D}).$$

Families of Prym varieties

Let $\pi : S \to T$ be a K3 double cover of another surface with *anti-symplectic* covering involution τ.

Thm (Nikulin): There exist 75 anti-symplectic involutions τ on K3s. The quotient $T = S/\tau$ is an Enriques or a rational surface.

A curve $D \subset T$ has a double cover $C \subset S$,

$$\begin{array}{ccc} C & \subset & S \\ {\scriptstyle 2:1}\downarrow & & {\scriptstyle 2:1}\downarrow \\ D & \subset & T. \end{array}$$

Let $\mathcal{D} \to |D|$ be the complete linear system in T, let $\widetilde{\mathcal{C}} \to |C|$ be the complete linear system in S, and let

$$\mathcal{C} := \pi^*\mathcal{D} \subset \widetilde{\mathcal{C}}.$$

Families of Prym varieties

There are two commuting *anti-symplectic* involutions on the Beauville-Mukai system $\overline{\mathrm{Jac}}^0(\tilde{\mathcal{C}}/|C|)$:

- the involution τ^* induced by τ,
- fibrewise duality $\mathcal{E} \mapsto \mathcal{E}xt^1_S(\mathcal{E}, \mathcal{O}(-C))$ (takes $\iota_*\mathcal{L} \mapsto \iota_*\mathcal{L}^\vee$).

Thm (Markushevich-Tikhomirov, Arbarello-Saccà-Ferretti, Matteini): We can construct a relative Prym variety

$$\mathrm{Prym}(\mathcal{C}/\mathcal{D}) := \mathrm{Fix}^0(\mathcal{E} \mapsto \mathcal{E}xt^1_S(\tau^*\mathcal{E}, \mathcal{O}(-C))) \subset \overline{\mathrm{Jac}}^0(\tilde{\mathcal{C}}/|C|).$$

This is a symplectic *variety* and a Lagrangian fibration over $|D|$.

$(1,2)$-polarized examples

3a. Markushevich-Tikhomirov system: S/T a K3 double cover of a degree two del Pezzo, C/D a genus three cover of an elliptic curve, $\mathrm{Prym}(C/D)$ an abelian surface of type $(1,2)$.

Then $\mathrm{Prym}(\mathcal{C}/\mathcal{D}) \to \mathbb{P}^2$ is an *irreducible* symplectic orbifold of dimension four, with 28 isolated $\mathbb{C}^4/\pm 1$ singularities.

Rmk: This orbifold is a partial resolution of the quotient of $\mathrm{Hilb}^2 S$ by a symplectic involution, sometimes called the *Nikulin variety*.

$(1,2)$-polarized examples

Another fibration on the Nikulin variety is constructed as follows.

3b. $F[2]$ acts fibrewise by translation on the Kummer K3

$$S \to E \times F/\pm 1 \to E/\pm 1 \cong \mathbb{P}^1$$

and there is an induced fibrewise action on $\mathrm{Hilb}^2 S \to \mathbb{P}^2$.

Rmk: Each element of $F[2]$ acts as a symplectic involution.

Quotient by the action of a single element and blow-up the K3 of singularities to get an orbifold X.

Prop: X is an isotrivial Lagrangian fibration over $\mathbb{P}^2$.

Rmk: Fibres $F \times F/\mathbb{Z}_2$ are $(1,2)$-polarized. Moreover, X has $b_2 = 16$, $b_3 = 0$, $b_4 = 178$, and 28 isolated $\mathbb{C}^4/\pm 1$ singularities.

Principally polarized examples

2a. Arbarello-Saccà-Ferretti system: S/T a K3 double cover of an Enriques, D genus $n+1$, $\mathrm{Prym}(C/D)$ principally polarized.

Then $\mathrm{Prym}(\mathcal{C}/\mathcal{D}) \to \mathbb{P}^n$ is a symplectic variety, which is

- birational to a Beauville-Mukai system if D is hyperelliptic,
- simply connected with no symplectic resolution otherwise,
- and irreducible if n is even.

If $n = 2$ or 3 it has isolated $\mathbb{C}^4/\pm 1$ or $\mathbb{C}^6/\pm 1$ singularities.

Lemma: If $C = C_1 \cup C_2$ with $C_1.C_2 = 2k$ then a neighbourhood of $[\mathcal{F}_1 \oplus \mathcal{F}_2] \in \mathrm{Prym}(\mathcal{C}/\mathcal{D})$ looks locally like $\mathbb{C}^{N-2k} \times (\mathbb{C}^{2k}/\pm 1)$.

Principally polarized examples

2b. Let S be a Kummer K3 surface with an elliptic fibration

$$S \longrightarrow E \times F / \pm 1 \longrightarrow E / \pm 1 \cong \mathbb{P}^1.$$

$\mathrm{Hilb}^2 S \to \mathbb{P}^2$ is an isotrivial fibration with smooth fibres $F \times F$.

The group $F[2] \cong \mathbb{Z}_2^{\oplus 2}$ acts by diagonal translation on $F \times F$ and fibrewise on $\mathrm{Hilb}^2 S$. Take the quotient $\mathrm{Hilb}^2 S / \mathbb{Z}_2^{\oplus 2}$ and blow-up codimension two singularities to get a symplectic orbifold X.

Prop: X is an isotrivial Lagrangian fibration over $\mathbb{P}^2$.

Rmk: Fibres $F \times F / F[2]$ are principally polarized. Moreover, X has $b_2 = 14$, $b_3 = 0$, $b_4 = 150$, and 36 isolated $\mathbb{C}^4 / \pm 1$ singularities.

Examples for $K_n(A)$

4a. Debarre system: Let $C \subset A$ give a polarization of type $(1, n+1)$. Then C has genus $n+2$ and $|C| \cong \mathbb{P}^n$. Let $\mathcal{C}/\mathbb{P}^n$ be the family of curves linearly equivalent to C and

$$Y := \overline{\mathrm{Jac}}^d(\mathcal{C}/\mathbb{P}^n) \longrightarrow \mathbb{P}^n.$$

We get a Lagrangian fibration

$$X := \mathsf{kernel}(\mathrm{Alb} : Y \longrightarrow A)$$

with $(1, \ldots, 1, n+1)$-polarized fibres $X_t = \ker(\overline{\mathrm{Jac}}^d C_t \longrightarrow A)$.

4b. If $A = E \times F$ then

$$\mathrm{Hilb}^{n+1} A \longrightarrow \mathrm{Sym}^{n+1} A \longrightarrow \mathrm{Sym}^{n+1} E \longrightarrow J^{n+1} E \cong E$$

induces an isotrivial Lagrangian fibration $K_n(A) \longrightarrow \mathbb{P}^n$ with $(1, \ldots, 1, n+1)$-polarized fibres

$$\cong \{(f_0, f_1, \ldots, f_n) \in F^{n+1} \mid f_0 + f_1 + \ldots + f_n = 0 \text{ in } F\}.$$

Summary of examples in four dimensions

Example	Polarization type
Beauville-Mukai system	$(1,1)$
$\mathrm{Hilb}^2 S$ of an elliptic K3 S	$(1,1)$
Arbarello-Saccà-Ferretti system	$(1,1)$
Isotrivial system on $\mathrm{Hilb}^2 S/\mathbb{Z}_2^{\oplus 2}$	$(1,1)$
Markushevich-Tikhomirov system	$(1,2)$
Isotrivial system on $\mathrm{Hilb}^2 S/\mathbb{Z}_2$	$(1,2)$
Debarre system for $A^{1,3}$	$(1,3)$
Isotrivial system on $K_2(E \times F)$	$(1,3)$

More six-dimensional examples

Matteini system: S/T a K3 double cover of a cubic del Pezzo, C/D a genus four cover of an elliptic curve, $\mathrm{Prym}(C/D)$ an abelian threefold of type $(1,1,2)$.

$\mathrm{Prym}(\mathcal{C}/\mathcal{D}) \to \mathbb{P}^3$ is an *irreducible* symplectic orbifold of dim$^{\mathrm{n}}$ six, with singularities that look like $\mathbb{C}^2 \times (\mathbb{C}^4/\pm 1)$ and $\mathbb{C}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$.

S-Shen system: S/T a K3 double cover of a degree one del Pezzo, $D \in |-2K_T|$, C/D a genus five cover of a genus two curve, $\mathrm{Prym}(C/D)$ an abelian threefold of type $(1,2,2)$. Then

$$\mathrm{Prym}(\mathcal{C}/\mathcal{D}) := \mathrm{Fix}^0(-) \subset \overline{\mathrm{Jac}}^0(\tilde{\mathcal{C}}/|C|) \leftarrow \mathrm{OG10}$$

is a symplectic variety of dim$^{\mathrm{n}}$ six, with 120 isolated singularities that look like $\mathbb{C}^6/\pm 1$.

A birational model

The del Pezzo T is a double cover of the quadric cone Q. The covering involution lifts to another anti-symplectic involution on S.

Their composition gives a symplectic involution on S, with quotient a singular K3 surface $\overline{S}$ with 8 A_1-singularities.

$$\begin{tikzcd}
 & C \subset S \arrow[dl] \arrow[d] \arrow[dr] & & \widetilde{C} \subset \widetilde{S} = \text{resolution of } \overline{S} \arrow[dl] \\
D \subset T \arrow[dr] & \mathbb{P}^2 \arrow[d] & \overline{C} \subset \overline{S} \arrow[dl] & \\
 & Q & &
\end{tikzcd}$$

A generic τ-invariant $C \subset S$ is an étale double cover of a genus three curve $\overline{C} \subset \overline{S}$, which is isomorphic to $\widetilde{C} \subset \widetilde{S}$.

A birational model

Pull-back induces a map

$$\mathrm{Jac}^0 \tilde{C} = \mathrm{Jac}^0 \overline{C} \longrightarrow \mathrm{Jac}^0 C$$

which is two-to-one onto its image $\mathrm{Prym}(C/D)$.

Let $\widetilde{\mathcal{M}} := \overline{\mathrm{Jac}}^0(\tilde{\mathcal{C}}/\mathbb{P}^3)$ be the Beauville-Mukai system of $\tilde{C} \subset \tilde{S}$. Then there is a rational dominant generically two-to-one map

$$\widetilde{\mathcal{M}} \dashrightarrow \mathrm{Prym}(\mathcal{C}/\mathcal{D}).$$

Moreover, $\widetilde{\mathcal{M}}$ is deformation equivalent to $\mathrm{Hilb}^3 \tilde{S}$.

Thm (S-Shen): $\mathrm{Prym}(\mathcal{C}/\mathcal{D})$ is a *primitive* symplectic variety:

- the symplectic structure is unique up to a scalar, $h^{2,0} = 1$,
- we have vanishing of the Hodge number $h^{1,0} = 0$.

Examples for $K_3(A)$ and OG6

Debarre system/OG6: Let $C \subset A$ give a polarization of type $(2,2)$. Then C has genus 5 and $|C| \cong \mathbb{P}^3$. Consider

$$Y := \overline{\mathrm{Jac}}^d(\mathcal{C}/\mathbb{P}^3) \longrightarrow \mathbb{P}^3,$$

i.e., $Y \cong M(0,[C],d-4)$ on A, and $X := \mathrm{kernel}(\mathrm{Alb} : Y \longrightarrow A)$.

- If d is odd then X is deformation equivalent to $K_3(A)$.
- If d is even then $\widetilde{X}$ is deformation equivalent to OG6.

Rmk: Both cases have fibres of polarization type $(1,2,2)$.

Summary of examples in six dimensions

Example	**Polarization type**
Beauville-Mukai system	$(1,1,1)$
$\mathrm{Hilb}^3 S$ of an elliptic K3 S	$(1,1,1)$
Arbarello-Saccà-Ferretti system	$(1,1,1)$
Matteini system	$(1,1,2)$
S-Shen system	$(1,2,2)$
Debarre system for $A^{1,4}$	$(1,1,4)$
Isotrivial system on $K_3(A)$	$(1,1,4)$
Debarre system for $A^{2,2}$	$(1,2,2)$
O'Grady 6 on $A^{2,2}$	$(1,2,2)$

Fibrations by Jacobians

Thm (Markushevich): Let $\mathcal{C}/\mathbb{P}^2$ be a flat family of integral Gorenstein curves of genus two such that $X = \overline{\mathrm{Jac}}^d(\mathcal{C}/\mathbb{P}^2)$ is a Lagrangian fibration (with X smooth!). Then $X \to \mathbb{P}^2$ must be a Beauville-Mukai integrable system.

Rmk: The general principally polarized abelian surface is the Jacobian of a genus two curve.

Thm (Matsushita): $R^i\pi_*\mathcal{O}_X \cong \Omega^i_{\mathbb{P}^2}$.

When $i = 1$ this says $TF \cong N^\vee_{F\subset X}$ for smooth fibres F.

Moreover, we have $R^1\pi_*\mathcal{O}_{\mathcal{C}} \cong R^1\pi_*\mathcal{O}_X \cong \Omega^1_{\mathbb{P}^2}$.

Fibrations by Jacobians

Proof: The relative canonical map gives a double cover

$$\mathcal{C} \longrightarrow \mathbb{P}(R^1\pi_*\mathcal{O}_{\mathcal{C}}) = \mathbb{P}(R^1\pi_*\mathcal{O}_X) = \mathbb{P}(\Omega^1_{\mathbb{P}^2}) \subset \mathbb{P}^2 \times (\mathbb{P}^2)^\vee$$

branched over the zero locus in $\mathbb{P}(\Omega^1_{\mathbb{P}^2})$ of a section of

$$\mathcal{O}_{\mathbb{P}(\Omega^1)}(6) \otimes \pi^*\mathcal{O}_{\mathbb{P}^2}(2k) = \mathcal{O}(2k+6,6)|_{\mathbb{P}(\Omega^1)}.$$

Now $R^1\pi_*\mathcal{O}_{\mathcal{C}} = \Omega^1_{\mathbb{P}^2}$ determines $k = -3$, so

$$\mathcal{O}(2k+6,6)|_{\mathbb{P}(\Omega^1)} = \mathcal{O}(0,6)|_{\mathbb{P}(\Omega^1)}$$

is pulled back from $(\mathbb{P}^2)^\vee$. This means the curves lie in the double cover of $(\mathbb{P}^2)^\vee$ branched over a sextic, i.e., a K3 surface.

$\square$

Fibrations by products of elliptic curves

Thm (Kamenova): Let $X \to \mathbb{P}^2$ be a Lagrangian fibration with

- X smooth,
- general fibre a product of two elliptic curves,
- "generic" singular fibres,
- and a global section.

Then X is birational to $\mathrm{Hilb}^2 S$ of an elliptic K3 surface S.

Thm (Debarre-Huybrechts-Macrì-Voisin): Let $X \to \mathbb{P}^2$ be a (numerical!) Lagrangian fibration with X smooth and a divisor $Y \subset X$ inducing a principal polarization on a general fibre. Then X is a deformation of $\mathrm{Hilb}^2 S$.

Rmk: These results cover examples 1a and 1b in four dimensions. Next consider example 3a with $(1,2)$-polarized fibres.

Fibrations by $(1,2)$-polarized fibres

If A is $(1,2)$-polarized then $A^\vee$ is too. Let $C \subset A^\vee$ be a polarization. Then C is genus three, and pull-back gives

$$A = \mathrm{Pic}^0 A^\vee \longrightarrow \mathrm{Jac}^0 C \longrightarrow E,$$

i.e., A is the Prym variety of a double cover $C \to E$.

Thm (Qin-S-): Let $\mathcal{C}/\mathcal{E}/\mathbb{P}^2$ be a flat family of double covers of reduced Gorenstein curves of genus three and one, respectively, such that $X = \overline{\mathrm{Prym}}(\mathcal{C}/\mathcal{E})$ is a Lagrangian fibration. Then $X \to \mathbb{P}^2$ must be a Markushevich-Tikhomirov system

Thus the elliptic curves $\mathcal{E}$ must lie in a degree two del Pezzo and the genus three curves $\mathcal{C}$ must lie in its K3 double cover.

Fibrations by $(1,2)$-polarized fibres

Proof: $f : \mathcal{C} \to \mathcal{E}$ is branched over a divisor of degree four on each fibre. Thus there is a line bundle $\mathcal{L}$ of degree two on each fibre with

$$f_*\mathcal{O}_\mathcal{C} = \mathcal{O}_\mathcal{E} \oplus \mathcal{L}^\vee.$$

Applying $R^1\pi_*$ gives (π always denotes projection to $\mathbb{P}^2$)

$$R^1\pi_*\mathcal{O}_\mathcal{C} = R^1\pi_*\mathcal{O}_\mathcal{E} \oplus R^1\pi_*\mathcal{L}^\vee.$$

On the other hand, $\mathrm{Jac}^0 C \sim \mathrm{Jac}^0 E \times \mathrm{Prym}(C/E)$ implies

$$\mathrm{H}^1(C, \mathcal{O}_C) = \mathrm{H}^1(E, \mathcal{O}_E) \oplus \mathrm{H}^1(X_t, \mathcal{O}_{X_t}),$$

$$R^1\pi_*\mathcal{O}_\mathcal{C} = R^1\pi_*\mathcal{O}_\mathcal{E} \oplus R^1\pi_*\mathcal{O}_X.$$

Fibrations by $(1,2)$-polarized fibres

Therefore

$$R^1\pi_*\mathcal{L}^\vee \cong R^1\pi_*\mathcal{O}_X \cong \Omega^1_{\mathbb{P}^2},$$

and we have a double cover

$$h : \mathcal{E} \longrightarrow \mathbb{P}(R^1\pi_*\mathcal{L}^\vee) = \mathbb{P}(\Omega^1_{\mathbb{P}^2}) \subset \mathbb{P}^2 \times (\mathbb{P}^2)^\vee$$

branched over the zero locus in $\mathbb{P}(\Omega^1_{\mathbb{P}^2})$ of a section of

$$\mathcal{O}_{\mathbb{P}(\Omega^1)}(4) \otimes \pi^*\mathcal{O}_{\mathbb{P}^2}(2d) = \mathcal{O}(2d+4,4)|_{\mathbb{P}(\Omega^1)}.$$

Recall $\mathcal{C} \to \mathcal{E}$ is branched over the zero locus in $\mathcal{E}$ of a section of

$$\mathcal{L}^2 \cong h^*(\mathcal{O}_{\mathbb{P}(\Omega^1)}(2) \otimes \pi^*\mathcal{O}_{\mathbb{P}^2}(e)) = h^*\mathcal{O}(e+2,2)|_{\mathbb{P}(\Omega^1)}.$$

Now $R^1\pi_*\mathcal{L}^\vee = \Omega^1_{\mathbb{P}^2}$ determines $d=-2$ and $e=-2$.

Fibrations by $(1,2)$-polarized fibres

So

$$\mathcal{O}(2d+4,4)|_{\mathbb{P}(\Omega^1)} = \mathcal{O}(0,4)|_{\mathbb{P}(\Omega^1)},$$
$$h^*\mathcal{O}(e+2,2)|_{\mathbb{P}(\Omega^1)} = h^*\mathcal{O}(0,2)|_{\mathbb{P}(\Omega^1)}$$

are pulled back from $(\mathbb{P}^2)^\vee$.

This means the elliptic curves $\mathcal{E}$ lie in the double cover of $(\mathbb{P}^2)^\vee$ branched over a quartic, i.e., a degree two del Pezzo surface T.

Rmk: If $g : T \to (\mathbb{P}^2)^\vee$ then

$$K_T \cong g^*(\mathcal{O}(-3)\otimes\mathcal{O}(2)) = g^*\mathcal{O}(-1).$$

Moreover, the genus three curves $\mathcal{C}$ lie in the double cover of T branched over the pull-back of a conic $\cong K_T^{-2}$, i.e., a K3 surface. $\square$

Fibrations by Jacobians in six dimensions

Thm (S-): Let $\mathcal{C}/\mathbb{P}^3$ be a flat family of reduced curves of genus 3 such that $X = \overline{\mathrm{Jac}}^d(\mathcal{C}/\mathbb{P}^3)$ is a Lagrangian fibration. If the curves

- are irreducible Gorenstein non-hyperelliptic, or
- are canonically positive 2-connected hyperelliptic,

then $X/\mathbb{P}^3$ must be a Beauville-Mukai integrable system.

Rmk: The general principally polarized abelian threefold is the Jacobian of a (non-hyperelliptic) curve of genus three.

Qu: If the general fibre of $X/\mathbb{P}^3$ is the product $E_1 \times E_2 \times E_3$ of elliptic curves, must X be $\mathrm{Hilb}^3 S$ of an elliptic K3 S?

Fibrations by Jacobians in six dimensions

Proof (in the non-hyperelliptic case): The relative canonical embedding gives

$$\mathcal{C} \to \mathbb{P}(R^1\pi_*\mathcal{O}_{\mathcal{C}}) = \mathbb{P}(R^1\pi_*\mathcal{O}_X) = \mathbb{P}(\Omega^1_{\mathbb{P}^3}) \subset \mathbb{P}^3 \times (\mathbb{P}^3)^\vee.$$

Indeed $\mathcal{C}$ is the zero locus in $\mathbb{P}(\Omega^1_{\mathbb{P}^3})$ of a section of

$$\mathcal{O}_{\mathbb{P}(\Omega^1)}(4) \otimes \pi^*\mathcal{O}_{\mathbb{P}^3}(k) = \mathcal{O}(k+4,4)|_{\mathbb{P}(\Omega^1)}.$$

Now $R^1\pi_*\mathcal{O}_{\mathcal{C}} = \Omega^1_{\mathbb{P}^3}$ determines $k = -4$, so

$$\mathcal{O}(k+4,4)|_{\mathbb{P}(\Omega^1)} = \mathcal{O}(0,4)|_{\mathbb{P}(\Omega^1)}$$

is pulled back from $(\mathbb{P}^3)^\vee$. This means the curves are hyperplane sections of a quartic K3 surface in $(\mathbb{P}^3)^\vee$.

An invariant of Lagrangian fibrations

Thm (Wieneck): In a family of Lagrangian fibrations the polarization type of the fibres is constant.

For fibrations on $\mathrm{Hilb}^n K3$ the polarization is *principal*

$$(d_1, d_2, \ldots, d_n) = (1, 1, \ldots, 1).$$

For fibrations on $K_n(A)$ the polarization is of type

$$(d_1, d_2, \ldots, d_{n-2}, d_{n-1}, d_n) = (1, 1, \ldots, 1, d_{n-1}, d_n)$$

with $d_{n-1} d_n = n + 1$.

Thm (Markman): If X is a general deformation of $\mathrm{Hilb}^n K3$ admitting a Lagrangian fibration then it is birational to a *Tate-Shafarevich twist* of Beauville-Mukai system.

Finiteness

Thm (S-): Fixing $d_1 | \dots | d_n$, there are finitely many Lagrangian fibrations up to deformation with

- polarization type $(d_1, \dots, d_n)$,
- a global section,
- maximally varying fibres,
- semistable singular fibres in codimension one.

Rmk: van Geemen-Voisin and Bakker proved that Lagrangian fibrations are maximally varying or *isotrivial*.

Rmk: Using a theorem of Charles, Kamenova showed that it is enough to assume there is a fibration with a fixed polarization type. (See also Debarre-Huybrechts-Macrì-Voisin.)

Thus, we want to bound the polarization type.

Restrictions on polarization type

Thm (S-): Let $X \to \mathbb{P}^2$ be a Lagrangian fibration (with X smooth!) and let $Y \in \mathrm{H}^2(X, \mathbb{Z})$ restrict to a polarization of type (d_1, d_2) on each smooth fibre. Then $d_1 d_2$ can take only the following twenty values up to a square:

$$1, 2, 3, 5, 7, 10, 15, 61, 62, 241, 246, 247,$$

$$249, 251, 253, 254, 255, 257, 258, 259$$

Example: Polarization types $(1, 6)$ and $(1, 11)$ are *not* possible. Debarre-Huybrechts-Macrì-Voisin showed that $(1, 2)$, $(1, 5)$, and $(1, 7)$ are also *not* possible (for X smooth!).

Rmk: In higher dimensions, the overall degree $d_1 \cdots d_n$ of the fibres can take only finitely many values up to an n^{th} power.

Restrictions on polarization type

Proof: Let L be the pullback of a hyperplane from $\mathbb{P}^2$. Then

$$\left(\int_X Y^2L^2\right)\left(\int_X c_2(\sigma\bar{\sigma})\right)^2 = \left(\int_X (\sigma\bar{\sigma})^2\right)\left(\int_X c_2 YL\right)^2$$

where $\int_X Y^2L^2 = 2!d_1d_2$ and Hitchin-S- formula gives

$$\frac{\left(\int_X c_2(\sigma\bar{\sigma})\right)^2}{\int_X (\sigma\bar{\sigma})^2} = \frac{24^2(2!)^2}{2^2}\sqrt{\hat{A}}[X].$$

Thus

$$1152d_1d_2\sqrt{\hat{A}}[X] = \left(\int_X c_2 YL\right)^2 \text{ is a square.}$$

Guan showed that $\sqrt{\hat{A}}[X]$ takes finitely many values.

$\square$

Summary of examples in four dimensions

Example	Polarization type
Beauville-Mukai system	$(1,1)$
$\mathrm{Hilb}^2 S$ of an elliptic K3 S	$(1,1)$
Arbarello-Saccà-Ferretti system	$(1,1)$
Isotrivial system on $\mathrm{Hilb}^2 S/\mathbb{Z}_2^{\oplus 2}$	$(1,1)$
Markushevich-Tikhomirov system	$(1,2)$
Isotrivial system on $\mathrm{Hilb}^2 S/\mathbb{Z}_2$	$(1,2)$
Debarre system for $A^{1,3}$	$(1,3)$
Isotrivial system on $K_2(E \times F)$	$(1,3)$

Summary of examples in six dimensions

Example	Polarization type
Beauville-Mukai system	$(1,1,1)$
$\mathrm{Hilb}^3 S$ of an elliptic K3 S	$(1,1,1)$
Arbarello-Saccà-Ferretti system	$(1,1,1)$
Matteini system	$(1,1,2)$
S-Shen system	$(1,2,2)$
Debarre system for $A^{1,4}$	$(1,1,4)$
Isotrivial system on $K_3(A)$	$(1,1,4)$
Debarre system for $A^{2,2}$	$(1,2,2)$
O'Grady 6 on $A^{2,2}$	$(1,2,2)$

Summary of examples in six dimensions with duals

Example	Type	Type	Dual
Beauville-Mukai	$(1,1,1)$	$(1,1,1)$	Beauville-Mukai
Hilb^3 of elliptic K3	$(1,1,1)$	$(1,1,1)$	Hilb^3 of elliptic K3
ASF system	$(1,1,1)$	$(1,1,1)$	ASF system
Matteini system	$(1,1,2)$		
S-Shen system	$(1,2,2)$	$(1,1,2)$	degenerate Matteini?
Debarre for $A^{1,4}$	$(1,1,4)$	$(1,4,4)$	Kim/S-
Isotrivial on $K_3(A)$	$(1,1,4)$	$(1,4,4)$	Kim/S-
Debarre for $A^{2,2}$	$(1,2,2)$	$(1,1,2)$	Kim/S-
O'Grady 6 on $A^{2,2}$	$(1,2,2)$	$(1,1,2)$	Kim/S-