Lagrangian fibrations in four and six dimensions

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Overview

- generalities on Lagrangian fibrations
- examples in four dimensions
- examples in six dimensions
- classification results
- polarization types

(partly joint work with Chen Shen and Xuqiang Qin)
Holomorphic symplectic manifolds

Let $X$ be a compact Kähler manifold with $c_1 = 0$.

**Thm (Bogomolov):** $\exists$ finite étale cover $\tilde{X}$ of $X$ with

$$\tilde{X} = T \times \prod_i CY_i \times \prod_j IHS_j,$$

$T =$ torus, $CY_i =$ (strict) Calabi-Yau manifolds, and $IHS_j =$ ...

**Def:** A compact Kähler manifold $X$ is a *holomorphic symplectic manifold* if it admits a non-degenerate holomorphic two-form $\sigma$.

In addition if $\pi_1(X) = 0$ and $H^0(\Omega^2)$ is generated by $\sigma$ then we say $X$ is an *irreducible holomorphic symplectic (IHS) manifold*. 
Examples of IHS manifolds

1. Hilbert schemes of points on K3 surfaces, $\text{Hilb}^n S \to \text{Sym}^n S$.

2. Generalized Kummer varieties, $\widetilde{\text{Hilb}}^{n+1} A = A \times K_n(A)$. Equivalently $K_n(A) := \text{kernel}(\text{Hilb}^{n+1} A \to \text{Sym}^{n+1} A \to A)$.


$$\text{Ext}^1(\mathcal{E}, \mathcal{E}) \times \text{Ext}^1(\mathcal{E}, \mathcal{E}) \to \text{Ext}^2(\mathcal{E}, \mathcal{E}) \xrightarrow{\text{tr}} H^2(O) \cong \mathbb{C}$$


Up to deformation, two/three (smooth) examples known in dimensions four/six: $\text{Hilb}^2 S$, $K_2(A)$, $\text{Hilb}^3 S$, $K_3(A)$, and OG6.
Let $X$ be an IHS manifold of dimension $2n$.

**Thm (Matsushita):** If $X \to B$ is a proper fibration then
1. $\dim B = n = \dim F$,
2. $F$ is Lagrangian wrt the holomorphic symplectic form $\sigma$,
3. generic fibre is a complex torus.

**Thm (Hwang):** $B$ is isomorphic to $\mathbb{P}^n$ if it is smooth.

**Thm (Huybrechts-Xu):** $B$ is smooth if $n = 2$, thus $B \cong \mathbb{P}^2$.

**Rmk (Voisin):** Hodge theory $\iff$ general fibre is an abelian variety.
Polarizations of abelian varieties

A polarization $H$ of an abelian variety gives

$$c_1(H) \in H^2(A, \mathbb{Z}) = \Lambda^2 H_1(A, \mathbb{Z})^*.$$ 

With respect to a standard basis

$$c_1(H) = \begin{bmatrix} d_2 & d_3 & \cdots & d_n \\ -d_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -d_n & -d_{n-1} & \cdots & 0 \end{bmatrix}$$

with $d_1 \mid d_2 \mid \cdots \mid d_n$. We call this the type of the polarization.
Examples for Hilb$^nS$

1a. Beauville-Mukai system: Let $C$ be a genus $n$ curve in a K3 surface $S$, with $|C| \cong \mathbb{P}^n$ and $C/\mathbb{P}^n$ the family of curves linearly equivalent to $C$.

$$X := \overline{\text{Jac}}^d(C/\mathbb{P}^n) \rightarrow \mathbb{P}^n$$

is a Lagrangian fibration, deformation equivalent to Hilb$^nS$.

Or $X \cong$ moduli space $M(0, [C], 1 - g + d)$ of stable sheaves on $S$.

1b. If $S \rightarrow \mathbb{P}^1$ is an elliptic K3 surface then the Hilbert scheme

$$\text{Hilb}^nS \rightarrow \text{Sym}^nS \rightarrow \text{Sym}^n\mathbb{P}^1 = \mathbb{P}^n$$

is a Lagrangian fibration. Its fibres look like

$$E_1 \times E_2 \times \cdots \times E_n.$$
(Generalized) Prym varieties

Let \( \pi : C \rightarrow D \) be a double cover of curves with covering involution \( \tau \). Then

\[
\text{Fix}^0(\tau^*) = \pi^* \text{Jac}^0 D \subset \text{Jac}^0 C.
\]

**Def:** The Prym variety of \( C/D \) is an abelian variety

\[
\text{Prym}(C/D) := \text{Fix}^0(-\tau^*),
\]

of dimension \( g_C - g_D \), principally polarized if \( \pi : C \rightarrow D \) has zero or two branch points, otherwise polarization type

\[
(1, \ldots, 1, 2, \ldots, 2).
\]

\[
g_C - 2g_D \quad g_D
\]
Families of Prym varieties

Let $\pi : S \to T$ be a K3 double cover of another surface with *anti-symplectic* covering involution $\tau$.

**Thm (Nikulin):** There exist 75 anti-symplectic involutions $\tau$ on K3s. The quotient $T = S/\tau$ is an Enriques or a rational surface.

A curve $D \subset T$ has a double cover $C \subset S$,

$$
\begin{array}{c c c}
C & \subset & S \\
\downarrow 2:1 & & \downarrow 2:1 \\
D & \subset & T.
\end{array}
$$

Let $\mathcal{D} \to |D|$ be the complete linear system in $T$, let $\tilde{\mathcal{C}} \to |C|$ be the complete linear system in $S$, and let

$$
\mathcal{C} := \pi^* \mathcal{D} \subset \tilde{\mathcal{C}}.
$$
There are two commuting anti-symplectic involutions on the Beauville-Mukai system $\overline{\text{Jac}}^0(\tilde{C}/|C|)$:

- the involution $\tau^*$ induced by $\tau$,
- fibrewise duality $\mathcal{E} \leftrightarrow \text{Ext}^1_S(\mathcal{E}, \mathcal{O}(-C))$ (takes $\iota_*\mathcal{L} \mapsto \iota_*\mathcal{L}^\vee$).

**Thm (Markushevich-Tikhomirov, Arbarello-Saccà-Ferretti, Matteini):** We can construct a relative Prym variety

$$\text{Prym}(\mathcal{C}/\mathcal{D}) := \text{Fix}^0(\mathcal{E} \leftrightarrow \text{Ext}^1_S(\tau^*\mathcal{E}, \mathcal{O}(-C))) \subset \overline{\text{Jac}}^0(\tilde{C}/|C|).$$

This is a symplectic variety and a Lagrangian fibration over $|D|$.
(1, 2)-polarized examples

3a. Markushevich-Tikhomirov system: \( S/T \) a K3 double cover of a degree two del Pezzo, \( C/D \) a genus three cover of an elliptic curve, \( \text{Prym}(C/D) \) an abelian surface of type \( (1, 2) \).

Then \( \text{Prym}(C/D) \to \mathbb{P}^2 \) is an irreducible symplectic orbifold of dimension four, with 28 isolated \( \mathbb{C}^4/\pm 1 \) singularities.

Rmk: This orbifold is a partial resolution of the quotient of \( \text{Hilb}^2 S \) by a symplectic involution, sometimes called the Nikulin variety.
(1, 2)-polarized examples

Another fibration on the Nikulin variety is constructed as follows.


$$S \rightarrow E \times F/\pm 1 \rightarrow E/\pm 1 \cong \mathbb{P}^1$$

and there is an induced fibrewise action on $\text{Hilb}^2 S \rightarrow \mathbb{P}^2$.


Quotient by the action of a single element and blow-up the K3 of singularities to get an orbifold $X$.

**Prop:** $X$ is an isotrivial Lagrangian fibration over $\mathbb{P}^2$.

**Rmk:** Fibres $F \times F/\mathbb{Z}_2$ are (1, 2)-polarized. Moreover, $X$ has $b_2 = 16$, $b_3 = 0$, $b_4 = 178$, and 28 isolated $\mathbb{C}^4/\pm 1$ singularities.
2a. **Arbarello-Saccà-Ferretti system:** $S/T$ a K3 double cover of an Enriques, $D$ genus $n + 1$, $\text{Prym}(C/D)$ principally polarized.

Then $\text{Prym}(C/D) \to \mathbb{P}^n$ is a symplectic variety, which is

- birational to a Beauville-Mukai system if $D$ is hyperelliptic,
- simply connected with no symplectic resolution otherwise,
- and irreducible if $n$ is even.

If $n = 2$ or 3 it has isolated $\mathbb{C}^4/\pm 1$ or $\mathbb{C}^6/\pm 1$ singularities.

**Lemma:** If $C = C_1 \cup C_2$ with $C_1.C_2 = 2k$ then a neighbourhood of $[\mathcal{F}_1 \oplus \mathcal{F}_2] \in \text{Prym}(C/D)$ looks locally like $\mathbb{C}^{N-2k} \times (\mathbb{C}^{2k}/\pm 1)$. 
Principally polarized examples

2b. Let $S$ be a Kummer K3 surface with an elliptic fibration

$$S \to E \times F/\pm 1 \to E/\pm 1 \cong \mathbb{P}^1.$$ 

Hilb$^2 S \to \mathbb{P}^2$ is an isotrivial fibration with smooth fibres $F \times F$.

The group $F[2] \cong \mathbb{Z}_2^{\oplus 2}$ acts by diagonal translation on $F \times F$ and fibrewise on Hilb$^2 S$. Take the quotient Hilb$^2 S/\mathbb{Z}_2^{\oplus 2}$ and blow-up codimension two singularities to get a symplectic orbifold $X$.

Prop: $X$ is an isotrivial Lagrangian fibration over $\mathbb{P}^2$.

Rmk: Fibres $F \times F/F[2]$ are principally polarized. Moreover, $X$ has $b_2 = 14$, $b_3 = 0$, $b_4 = 150$, and 36 isolated $\mathbb{C}^4/\pm 1$ singularities.
Examples for $K_n(A)$

4a. **Debarre system:** Let $C \subset A$ give a polarization of type $(1, n + 1)$. Then $C$ has genus $n + 2$ and $|C| \cong \mathbb{P}^n$. Let $C/\mathbb{P}^n$ be the family of curves linearly equivalent to $C$ and

$$Y := \overline{\text{Jac}}^d(C/\mathbb{P}^n) \longrightarrow \mathbb{P}^n.$$ 

We get a Lagrangian fibration

$$X := \text{kernel}(\text{Alb} : Y \longrightarrow A)$$

with $(1, \ldots, 1, n + 1)$-polarized fibres $X_t = \text{ker}(\overline{\text{Jac}}^d C_t \longrightarrow A)$.

4b. If $A = E \times F$ then

$$\text{Hilb}^{n+1} A \longrightarrow \text{Sym}^{n+1} A \longrightarrow \text{Sym}^{n+1} E \longrightarrow J^{n+1} E \cong E$$

induces an isotrivial Lagrangian fibration $K_n(A) \longrightarrow \mathbb{P}^n$ with $(1, \ldots, 1, n + 1)$-polarized fibres

$$\cong \{(f_0, f_1, \ldots, f_n) \in F^{n+1} \mid f_0 + f_1 + \ldots + f_n = 0 \text{ in } F\}.$$
Summary of examples in four dimensions

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<th>Polarization type</th>
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**More six-dimensional examples**

**Matteini system:** $S/T$ a K3 double cover of a cubic del Pezzo, $C/D$ a genus four cover of an elliptic curve, $\text{Prym}(C/D)$ an abelian threefold of type $(1,1,2)$.

$\text{Prym}(C/D) \to \mathbb{P}^3$ is an *irreducible* symplectic orbifold of dim $n$ six, with singularities that look like $\mathbb{C}^2 \times (\mathbb{C}^4/\pm 1)$ and $\mathbb{C}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$.

**S-Shen system:** $S/T$ a K3 double cover of a degree one del Pezzo, $D \in |-2K_T|$, $C/D$ a genus five cover of a genus two curve, $\text{Prym}(C/D)$ an abelian threefold of type $(1,2,2)$. Then

$$\text{Prym}(C/D) := \text{Fix}^0(-) \subset \text{Jac}^0(\tilde{C}/|C|) \leftarrow \text{OG}10$$

is a symplectic variety of dim $n$ six, with 120 isolated singularities that look like $\mathbb{C}^6/\pm 1$. 
A birational model

The del Pezzo $T$ is a double cover of the quadric cone $Q$. The covering involution lifts to another anti-symplectic involution on $S$. Their composition gives a symplectic involution on $S$, with quotient a singular K3 surface $\bar{S}$ with 8 $A_1$-singularities.

A generic $\tau$-invariant $C \subset S$ is an étale double cover of a genus three curve $\bar{C} \subset \bar{S}$, which is isomorphic to $\tilde{C} \subset \tilde{S}$. 
A birational model

Pull-back induces a map

$$\text{Jac}^0 \tilde{C} = \text{Jac}^0 \overline{C} \longrightarrow \text{Jac}^0 C$$

which is two-to-one onto its image $\text{Prym}(C/D)$.

Let $\tilde{M} := \text{Jac}^0 (\tilde{C}/\mathbb{P}^3)$ be the Beauville-Mukai system of $\tilde{C} \subset \tilde{S}$. Then there is a rational dominant generically two-to-one map

$$\tilde{M} \longrightarrow \text{Prym}(C/D).$$

Moreover, $\tilde{M}$ is deformation equivalent to $\text{Hilb}^3 \tilde{S}$.

**Thm (S-Shen):** $\text{Prym}(C/D)$ is a *primitive* symplectic variety:

- the symplectic structure is unique up to a scalar, $h^{2,0} = 1$,
- we have vanishing of the Hodge number $h^{1,0} = 0$. 
Debarre system/OG6: Let $C \subset A$ give a polarization of type $(2, 2)$. Then $C$ has genus 5 and $|C| \cong \mathbb{P}^3$. Consider

$$Y := \overline{\text{Jac}}^d(C/\mathbb{P}^3) \to \mathbb{P}^3,$$

i.e., $Y \cong M(0, [C], d - 4)$ on $A$, and $X := \text{kernel}(\text{Alb} : Y \to A)$.

- If $d$ is odd then $X$ is deformation equivalent to $K_3(A)$.
- If $d$ is even then $\tilde{X}$ is deformation equivalent to OG6.

Rmk: Both cases have fibres of polarization type $(1, 2, 2)$. 
## Summary of examples in six dimensions

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**Lagrangian fibrations**

**4d examples**

**6d examples**

**classification**

**polarizations**

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**Fibrations by Jacobians**

**Thm (Markushevich):** Let \( C/\mathbb{P}^2 \) be a flat family of integral Gorenstein curves of genus two such that \( X = \Jac^d(C/\mathbb{P}^2) \) is a Lagrangian fibration (with \( X \) smooth!). Then \( X \to \mathbb{P}^2 \) must be a Beauville-Mukai integrable system.

**Rmk:** The general principally polarized abelian surface is the Jacobian of a genus two curve.

**Thm (Matsushita):** \( R^i\pi_*\mathcal{O}_X \cong \Omega^i_{\mathbb{P}^2} \).

When \( i = 1 \) this says \( TF \cong N^\vee_{F\subset X} \) for smooth fibres \( F \).

Moreover, we have \( R^1\pi_*\mathcal{O}_C \cong R^1\pi_*\mathcal{O}_X \cong \Omega^1_{\mathbb{P}^2} \).
Fibrations by Jacobians

**Proof:** The relative canonical map gives a double cover

\[ C \longrightarrow \mathbb{P}(R^1\pi_*\mathcal{O}_C) = \mathbb{P}(R^1\pi_*\mathcal{O}_X) = \mathbb{P}(\Omega_{\mathbb{P}^2}^1) \subset \mathbb{P}^2 \times (\mathbb{P}^2)^{\vee} \]

branched over the zero locus in \( \mathbb{P}(\Omega_{\mathbb{P}^2}^1) \) of a section of

\[ \mathcal{O}_{\mathbb{P}(\Omega^1)}(6) \otimes \pi^*\mathcal{O}_{\mathbb{P}^2}(2k) = \mathcal{O}(2k + 6, 6)|_{\mathbb{P}(\Omega^1)}. \]

Now \( R^1\pi_*\mathcal{O}_C = \Omega_{\mathbb{P}^2}^1 \) determines \( k = -3 \), so

\[ \mathcal{O}(2k + 6, 6)|_{\mathbb{P}(\Omega^1)} = \mathcal{O}(0, 6)|_{\mathbb{P}(\Omega^1)} \]

is pulled back from \( (\mathbb{P}^2)^{\vee} \). This means the curves lie in the double cover of \( (\mathbb{P}^2)^{\vee} \) branched over a sextic, i.e., a K3 surface.

\[ \square \]
Fibrations by products of elliptic curves

**Thm (Kamenova):** Let $X \to \mathbb{P}^2$ be a Lagrangian fibration with

- $X$ smooth,
- general fibre a product of two elliptic curves,
- “generic” singular fibres,
- and a global section.

Then $X$ is birational to $\text{Hilb}^2 S$ of an elliptic $K3$ surface $S$.

**Thm (Debarre-Huybrechts-Macrì-Voisin):** Let $X \to \mathbb{P}^2$ be a (numerical!) Lagrangian fibration with $X$ smooth and a divisor $Y \subset X$ inducing a principal polarization on a general fibre. Then $X$ is a deformation of $\text{Hilb}^2 S$.

**Rmk:** These results cover examples 1a and 1b in four dimensions. Next consider example 3a with $(1, 2)$-polarized fibres.
Fibrations by $(1, 2)$-polarized fibres

If $A$ is $(1, 2)$-polarized then $A^\vee$ is too. Let $C \subset A^\vee$ be a polarization. Then $C$ is genus three, and pull-back gives

$$A = \text{Pic}^0 A^\vee \longrightarrow \text{Jac}^0 C \longrightarrow E,$$

i.e., $A$ is the Prym variety of a double cover $C \to E$.

**Thm (Qin-S-):** Let $\mathcal{C}/\mathcal{E}/\mathbb{P}^2$ be a flat family of double covers of reduced Gorenstein curves of genus three and one, respectively, such that $X = \overline{\text{Prym}}(\mathcal{C}/\mathcal{E})$ is a Lagrangian fibration. Then $X \to \mathbb{P}^2$ must be a Markushevich-Tikhomirov system

Thus the elliptic curves $\mathcal{E}$ must lie in a degree two del Pezzo and the genus three curves $C$ must lie in its K3 double cover.
Fibrations by \((1, 2)\)-polarized fibres

**Proof:** \(f : C \to \mathcal{E}\) is branched over a divisor of degree four on each fibre. Thus there is a line bundle \(\mathcal{L}\) of degree two on each fibre with

\[
f_*\mathcal{O}_C = \mathcal{O}_\mathcal{E} \oplus \mathcal{L}^\vee.
\]

Applying \(R^1\pi_*\) gives (\(\pi\) always denotes projection to \(\mathbb{P}^2\))

\[
R^1\pi_*\mathcal{O}_C = R^1\pi_*\mathcal{O}_\mathcal{E} \oplus R^1\pi_*\mathcal{L}^\vee.
\]

On the other hand, \(\text{Jac}^0 C \sim \text{Jac}^0 E \times \text{Prym}(C/E)\) implies

\[
H^1(C, \mathcal{O}_C) = H^1(E, \mathcal{O}_E) \oplus H^1(X_t, \mathcal{O}_{X_t}),
\]

\[
R^1\pi_*\mathcal{O}_C = R^1\pi_*\mathcal{O}_\mathcal{E} \oplus R^1\pi_*\mathcal{O}_X.
\]
Fibrations by \((1, 2)\)-polarized fibres

Therefore

\[ R^1\pi_* \mathcal{L}^\vee \cong R^1\pi_* \mathcal{O}_X \cong \Omega^1_{\mathbb{P}^2}, \]

and we have a double cover

\[ h : \mathcal{E} \longrightarrow \mathbb{P}(R^1\pi_* \mathcal{L}^\vee) = \mathbb{P}(\Omega^1_{\mathbb{P}^2}) \subset \mathbb{P}^2 \times (\mathbb{P}^2)^\vee \]

branched over the zero locus in \(\mathbb{P}(\Omega^1_{\mathbb{P}^2})\) of a section of

\[ \mathcal{O}_{\mathbb{P}(\Omega^1)}(4) \otimes \pi^* \mathcal{O}_{\mathbb{P}^2}(2d) = \mathcal{O}(2d + 4, 4)|_{\mathbb{P}(\Omega^1)}. \]

Recall \(\mathcal{C} \to \mathcal{E}\) is branched over the zero locus in \(\mathcal{E}\) of a section of

\[ \mathcal{L}^2 \cong h^*(\mathcal{O}_{\mathbb{P}(\Omega^1)}(2) \otimes \pi^* \mathcal{O}_{\mathbb{P}^2}(e)) = h^* \mathcal{O}(e + 2, 2)|_{\mathbb{P}(\Omega^1)}. \]

Now \(R^1\pi_* \mathcal{L}^\vee \cong \Omega^1_{\mathbb{P}^2}\) determines \(d = -2\) and \(e = -2\).
Fibrations by \((1, 2)\)-polarized fibres

So

\[
\mathcal{O}(2d + 4, 4)|_{\mathbb{P}(\Omega^1)} = \mathcal{O}(0, 4)|_{\mathbb{P}(\Omega^1)},
\]

\[
h^*\mathcal{O}(e + 2, 2)|_{\mathbb{P}(\Omega^1)} = h^*\mathcal{O}(0, 2)|_{\mathbb{P}(\Omega^1)}
\]

are pulled back from \((\mathbb{P}^2)^\vee\).

This means the elliptic curves \(\mathcal{E}\) lie in the double cover of \((\mathbb{P}^2)^\vee\) branched over a quartic, i.e., a degree two del Pezzo surface \(T\).

**Rmk:** If \(g : T \rightarrow (\mathbb{P}^2)^\vee\) then

\[
K_T \cong g^*(\mathcal{O}(-3) \otimes \mathcal{O}(2)) = g^*\mathcal{O}(-1).
\]

Moreover, the genus three curves \(\mathcal{C}\) lie in the double cover of \(T\) branched over the pull-back of a conic \(\cong K_T^{-2}\), i.e., a K3 surface.
Fibrations by Jacobians in six dimensions

**Thm (S-):** Let \( C/\mathbb{P}^3 \) be a flat family of reduced curves of genus 3 such that \( X = \overline{\text{Jac}}^d(C/\mathbb{P}^3) \) is a Lagrangian fibration. If the curves
- are irreducible Gorenstein non-hyperelliptic, or
- are canonically positive 2-connected hyperelliptic,
then \( X/\mathbb{P}^3 \) must be a Beauville-Mukai integrable system.

**Rmk:** The general principally polarized abelian threefold is the Jacobian of a (non-hyperelliptic) curve of genus three.

**Qu:** If the general fibre of \( X/\mathbb{P}^3 \) is the product \( E_1 \times E_2 \times E_3 \) of elliptic curves, must \( X \) be \( \text{Hilb}^3 S \) of an elliptic K3 \( S \)?
Fibrations by Jacobians in six dimensions

Proof (in the non-hyperelliptic case): The relative canonical embedding gives

\[ \mathcal{C} \to \mathbb{P}(R^1\pi_*\mathcal{O}_C) = \mathbb{P}(R^1\pi_*\mathcal{O}_X) = \mathbb{P}(\Omega^1_{\mathbb{P}^3}) \subset \mathbb{P}^3 \times (\mathbb{P}^3)^\vee. \]

Indeed \( \mathcal{C} \) is the zero locus in \( \mathbb{P}(\Omega^1_{\mathbb{P}^3}) \) of a section of

\[ \mathcal{O}_{\mathbb{P}(\Omega^1)}(4) \otimes \pi^*\mathcal{O}_{\mathbb{P}^3}(k) = \mathcal{O}(k + 4, 4)|_{\mathbb{P}(\Omega^1)}. \]

Now \( R^1\pi_*\mathcal{O}_C = \Omega^1_{\mathbb{P}^3} \) determines \( k = -4 \), so

\[ \mathcal{O}(k + 4, 4)|_{\mathbb{P}(\Omega^1)} = \mathcal{O}(0, 4)|_{\mathbb{P}(\Omega^1)} \]

is pulled back from \( (\mathbb{P}^3)^\vee \). This means the curves are hyperplane sections of a quartic K3 surface in \( (\mathbb{P}^3)^\vee \).
An invariant of Lagrangian fibrations

**Thm (Wieneck):** In a family of Lagrangian fibrations the polarization type of the fibres is constant.

For fibrations on $\text{Hilb}^nK3$ the polarization is *principal*

$$(d_1, d_2, \ldots, d_n) = (1, 1, \ldots, 1).$$

For fibrations on $K_n(A)$ the polarization is of type

$$(d_1, d_2, \ldots, d_{n-2}, d_{n-1}, d_n) = (1, 1, \ldots, 1, d_{n-1}, d_n)$$

with $d_{n-1}d_n = n + 1$.

**Thm (Markman):** If $X$ is a general deformation of $\text{Hilb}^nK3$ admitting a Lagrangian fibration then it is birational to a *Tate-Shafarevich twist* of Beauville-Mukai system.
Finiteness

**Thm (S-):** Fixing $d_1 \ldots | d_n$, there are finitely many Lagrangian fibrations up to deformation with

- polarization type $(d_1, \ldots, d_n)$,
- a global section,
- maximally varying fibres,
- semistable singular fibres in codimension one.

**Rmk:** van Geemen-Voisin and Bakker proved that Lagrangian fibrations are maximally varying or *isotrivial*.

**Rmk:** Using a theorem of Charles, Kamenova showed that it is enough to assume there is a fibration with a fixed polarization type. (See also Debarre-Huybrechts-Macrì-Voisin.)

*Thus, we want to bound the polarization type.*
Restrictions on polarization type

**Thm (S-):** Let $X \to \mathbb{P}^2$ be a Lagrangian fibration (with $X$ smooth!) and let $Y \in H^2(X, \mathbb{Z})$ restrict to a polarization of type $(d_1, d_2)$ on each smooth fibre. Then $d_1 d_2$ can take only the following twenty values up to a square:

$$1, 2, 3, 5, 7, 10, 15, 61, 62, 241, 246, 247, 249, 251, 253, 254, 255, 257, 258, 259$$

**Example:** Polarization types $(1, 6)$ and $(1, 11)$ are *not* possible. Debarre-Huybrechts-Macrì-Voisin showed that $(1, 2)$, $(1, 5)$, and $(1, 7)$ are also *not* possible (for $X$ smooth!).

**Rmk:** In higher dimensions, the overall degree $d_1 \cdots d_n$ of the fibres can take only finitely many values up to an $n^{th}$ power.
Restrictions on polarization type

**Proof:** Let $L$ be the pullback of a hyperplane from $\mathbb{P}^2$. Then

$$\left(\int_X Y^2 L^2\right) \left(\int_X c_2(\sigma \bar{\sigma})\right)^2 = \left(\int_X (\sigma \bar{\sigma})^2\right) \left(\int_X c_2 Y L\right)^2$$

where $\int_X Y^2 L^2 = 2!d_1d_2$ and Hitchin-S- formula gives

$$\frac{\left(\int_X c_2(\sigma \bar{\sigma})\right)^2}{\int_X (\sigma \bar{\sigma})^2} = \frac{24^2(2!)^2}{2^2} \sqrt{\hat{A}[X]}.$$

Thus

$$1152d_1d_2\sqrt{\hat{A}[X]} = \left(\int_X c_2 Y L\right)^2$$

is a square.

Guan showed that $\sqrt{\hat{A}[X]}$ takes finitely many values.
Summary of examples in four dimensions

<table>
<thead>
<tr>
<th>Example</th>
<th>Polarization type</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beauville-Mukai system</td>
<td>$(1, 1)$</td>
</tr>
<tr>
<td>Hilb$^2 S$ of an elliptic K3 $S$</td>
<td>$(1, 1)$</td>
</tr>
<tr>
<td>Arbarello-Saccà-Ferretti system</td>
<td>$(1, 1)$</td>
</tr>
<tr>
<td>Isotrivial system on Hilb$^2 S/\mathbb{Z}_2^2$</td>
<td>$(1, 1)$</td>
</tr>
<tr>
<td>Markushevich-Tikhomirov system</td>
<td>$(1, 2)$</td>
</tr>
<tr>
<td>Isotrivial system on Hilb$^2 S/\mathbb{Z}_2$</td>
<td>$(1, 2)$</td>
</tr>
<tr>
<td>Debarre system for $A^{1,3}$</td>
<td>$(1, 3)$</td>
</tr>
<tr>
<td>Isotrivial system on $K_2(E \times F)$</td>
<td>$(1, 3)$</td>
</tr>
</tbody>
</table>
# Summary of examples in six dimensions

<table>
<thead>
<tr>
<th>Example</th>
<th>Polarization type</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beauville-Mukai system</td>
<td>(1, 1, 1)</td>
</tr>
<tr>
<td>Hilb$^3 S$ of an elliptic K3 S</td>
<td>(1, 1, 1)</td>
</tr>
<tr>
<td>Arbarello-Saccà-Ferretti system</td>
<td>(1, 1, 1)</td>
</tr>
<tr>
<td>Matteini system</td>
<td>(1, 1, 2)</td>
</tr>
<tr>
<td>S-Shen system</td>
<td>(1, 2, 2)</td>
</tr>
<tr>
<td>Debarre system for $A^{1,4}$</td>
<td>(1, 1, 4)</td>
</tr>
<tr>
<td>Isotrivial system on $K_3(A)$</td>
<td>(1, 1, 4)</td>
</tr>
<tr>
<td>Debarre system for $A^{2,2}$</td>
<td>(1, 2, 2)</td>
</tr>
<tr>
<td>O'Grady 6 on $A^{2,2}$</td>
<td>(1, 2, 2)</td>
</tr>
</tbody>
</table>
### Summary of examples in six dimensions with duals

<table>
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<th>Type 2</th>
<th>Dual</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beauville-Mukai</td>
<td>$p^{1,1,1}$</td>
<td>$p^{1,1,1}$</td>
<td>Beauville-Mukai</td>
</tr>
<tr>
<td>Hilb$^3$ of elliptic K3</td>
<td>$p^{1,1,1}$</td>
<td>$p^{1,1,1}$</td>
<td>Hilb$^3$ of elliptic K3</td>
</tr>
<tr>
<td>ASF system</td>
<td>$p^{1,1,1}$</td>
<td>$p^{1,1,1}$</td>
<td>ASF system</td>
</tr>
<tr>
<td>Matteini system</td>
<td>$p^{1,1,2}$</td>
<td>$p^{1,1,1}$</td>
<td></td>
</tr>
<tr>
<td>S-Shen system</td>
<td>$p^{1,2,2}$</td>
<td>$p^{1,1,2}$</td>
<td>degenerate Matteini?</td>
</tr>
<tr>
<td>Debarre for $A^{1,4}$</td>
<td>$p^{1,1,4}$</td>
<td>$p^{1,4,4}$</td>
<td>Kim/S-</td>
</tr>
<tr>
<td>Isotrivial on $K_3(A)$</td>
<td>$p^{1,1,4}$</td>
<td>$p^{1,4,4}$</td>
<td>Kim/S-</td>
</tr>
<tr>
<td>Debarre for $A^{2,2}$</td>
<td>$p^{1,2,2}$</td>
<td>$p^{1,1,2}$</td>
<td>Kim/S-</td>
</tr>
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</table>