Lagrangian fibrations Part I : four dimensions

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Overview

- generalities on Lagrangian fibrations
- examples in four dimensions
- classification results
- polarization types

(partly joint work with Xuqiang Qin)

Holomorphic symplectic manifolds

Let X be a compact Kähler manifold with $c_1 = 0$.

Thm (Bogomolov): $\exists$ finite étale cover $\widetilde{X}$ of X with

$$\widetilde{X} = T \times \prod_i CY_i \times \prod_j IHS_j,$$

T = torus, CY_i = (strict) Calabi-Yau manifolds, and $IHS_j = \ldots$

Def: A compact Kähler manifold X is a *holomorphic symplectic manifold* if it admits a non-degenerate holomorphic two-form σ.

In addition if $\pi_1(X) = 0$ and $\mathrm{H}^0(\Omega^2)$ is generated by σ then we say X is an *irreducible holomorphic symplectic (IHS) manifold*.

Examples of IHS manifolds

1. Hilbert schemes of points on K3 surfaces, $\mathrm{Hilb}^n S \to \mathrm{Sym}^n S$.

2. Generalized Kummer varieties, $\widetilde{\mathrm{Hilb}^{n+1}} A = A \times K_n(A)$.
Equivalently $K_n(A) := \text{kernel}(\mathrm{Hilb}^{n+1} A \longrightarrow \mathrm{Sym}^{n+1} A \longrightarrow A)$.

3. Mukai moduli spaces of stable sheaves on K3/abelian surfaces.

$$\mathrm{Ext}^1(\mathcal{E},\mathcal{E}) \times \mathrm{Ext}^1(\mathcal{E},\mathcal{E}) \to \mathrm{Ext}^2(\mathcal{E},\mathcal{E}) \xrightarrow{\mathrm{tr}} \mathrm{H}^2(\mathcal{O}) \cong \mathbb{C}$$

4. O'Grady's spaces, OG6 and OG10.

Up to deformation, two (smooth) examples known in dimension four: $\mathrm{Hilb}^2 S$ and $K_2(A)$.

Fibrations

Let X be an IHS manifold of dimension $2n$.

Thm (Matsushita): If $X \to B$ is a proper fibration then

1. $\dim B = n = \dim F$,
2. F is Lagrangian wrt the holomorphic symplectic form σ,
3. generic fibre is a complex torus.

Thm (Hwang): B is isomorphic to $\mathbb{P}^n$ if it is smooth.

Thm (Huybrechts-Xu): B is smooth if $n = 2$, thus $B \cong \mathbb{P}^2$.

Rmk (Voisin): Hodge theory $\implies$ general fibre is an abelian variety.

Polarizations of abelian varieties

A polarization H of an abelian variety gives

$$c_1(H) \in \mathrm{H}^2(A,\mathbb{Z}) = \Lambda^2\mathrm{H}_1(A,\mathbb{Z})^*.$$

With respect to a standard basis

$$c_1(H) = \begin{bmatrix} & & & & d_1 & & & \\ & 0 & & & & d_2 & & \\ & & & & & & \ddots & \\ & & & & & & & d_n \\ -d_1 & & & & & & & \\ & -d_2 & & & & & 0 & \\ & & \ddots & & & & & \\ & & & -d_n & & & & \end{bmatrix}$$

with $d_1|d_2|\cdots|d_n$. We call this the *type* of the polarization.

Examples for $\mathrm{Hilb}^2 S$

1a. Beauville-Mukai system: Let C be a genus n curve in a K3 surface S, with $|C| \cong \mathbb{P}^n$ and $\mathcal{C}/\mathbb{P}^n$ the family of curves linearly equivalent to C.

$$X := \overline{\mathrm{Jac}}^d(\mathcal{C}/\mathbb{P}^n) \longrightarrow \mathbb{P}^n$$

is a Lagrangian fibration, deformation equivalent to $\mathrm{Hilb}^n S$.

Or $X \cong$ moduli space $M(0, [C], 1 - g + d)$ of stable sheaves on S.

1b. If $S \longrightarrow \mathbb{P}^1$ is an elliptic K3 surface then the Hilbert scheme

$$\mathrm{Hilb}^n S \to \mathrm{Sym}^n S \to \mathrm{Sym}^n \mathbb{P}^1 = \mathbb{P}^n$$

is a Lagrangian fibration. Its fibres look like

$$E_1 \times E_2 \times \cdots \times E_n.$$

(Generalized) Prym varieties

Let $\pi : C \to D$ be a double cover of curves with covering involution τ. Then

$$\mathrm{Fix}^0(\tau^*) = \pi^* \mathrm{Jac}^0 D \subset \mathrm{Jac}^0 C.$$

Def: The Prym variety of C/D is an abelian variety

$$\mathrm{Prym}(C/D) := \mathrm{Fix}^0(-\tau^*),$$

of dimension $g_C - g_D$, principally polarized if $\pi : C \to D$ has zero or two branch points, otherwise polarization type

$$(\underbrace{1, \ldots, 1}_{g_C - 2g_D}, \underbrace{2, \ldots, 2}_{g_D}).$$

Families of Prym varieties

Let $\pi : S \to T$ be a K3 double cover of another surface with *anti-symplectic* covering involution τ.

Thm (Nikulin): There exist 75 anti-symplectic involutions τ on K3s. The quotient $T = S/\tau$ is an Enriques or a rational surface.

A curve $D \subset T$ has a double cover $C \subset S$,

$$\begin{array}{ccc} C & \subset & S \\ {\scriptstyle 2:1}\downarrow & & {\scriptstyle 2:1}\downarrow \\ D & \subset & T. \end{array}$$

Let $\mathcal{D} \to |D|$ be the complete linear system in T, let $\widetilde{\mathcal{C}} \to |C|$ be the complete linear system in S, and let

$$\mathcal{C} := \pi^* \mathcal{D} \subset \widetilde{\mathcal{C}}.$$

Families of Prym varieties

There are two commuting *anti-symplectic* involutions on the Beauville-Mukai system $\overline{\mathrm{Jac}}^0(\widetilde{\mathcal{C}}/|C|)$:

- the involution τ^* induced by τ,
- fibrewise duality $\mathcal{E} \mapsto \mathcal{E}xt^1_S(\mathcal{E}, \mathcal{O}(-C))$ (takes $\iota_*\mathcal{L} \mapsto \iota_*\mathcal{L}^\vee$).

Thm (Markushevich-Tikhomirov, Arbarello-Saccà-Ferretti, Matteini): We can construct a relative Prym variety

$$\mathrm{Prym}(\mathcal{C}/\mathcal{D}) := \mathrm{Fix}^0(\mathcal{E} \mapsto \mathcal{E}xt^1_S(\tau^*\mathcal{E}, \mathcal{O}(-C))) \subset \overline{\mathrm{Jac}}^0(\widetilde{\mathcal{C}}/|C|).$$

This is a symplectic *variety* and a Lagrangian fibration over $|D|$.

$(1,2)$-polarized examples

3a. Markushevich-Tikhomirov system: S/T a K3 double cover of a degree two del Pezzo, C/D a genus three cover of an elliptic curve, $\mathrm{Prym}(C/D)$ an abelian surface of type $(1,2)$.

Then $\mathrm{Prym}(\mathcal{C}/\mathcal{D}) \to \mathbb{P}^2$ is an *irreducible* symplectic orbifold of dimension four, with 28 isolated $\mathbb{C}^4/\pm 1$ singularities.

Rmk: This orbifold is a partial resolution of the quotient of $\mathrm{Hilb}^2 S$ by a symplectic involution, sometimes called the *Nikulin variety*.

$(1,2)$-polarized examples

Another fibration on the Nikulin variety is constructed as follows.

3b. $F[2]$ acts fibrewise by translation on the Kummer K3

$$S \to E \times F/\pm 1 \to E/\pm 1 \cong \mathbb{P}^1$$

and there is an induced fibrewise action on $\mathrm{Hilb}^2 S \to \mathbb{P}^2$.

Rmk: Each element of $F[2]$ acts as a symplectic involution.

Quotient by the action of a single element and blow-up the K3 of singularities to get an orbifold X.

Prop: X is an isotrivial Lagrangian fibration over $\mathbb{P}^2$.

Rmk: Fibres $F \times F/\mathbb{Z}_2$ are $(1,2)$-polarized. Moreover, X has $b_2 = 16$, $b_3 = 0$, $b_4 = 178$, and 28 isolated $\mathbb{C}^4/\pm 1$ singularities.

Principally polarized examples

2a. Arbarello-Saccà-Ferretti system: S/T a K3 double cover of an Enriques, D genus $n+1$, $\mathrm{Prym}(C/D)$ principally polarized.

Then $\mathrm{Prym}(\mathcal{C}/\mathcal{D}) \to \mathbb{P}^n$ is a symplectic variety, which is

- birational to a Beauville-Mukai system if D is hyperelliptic,
- simply connected with no symplectic resolution otherwise,
- and irreducible if n is even.

If $n = 2$ it has isolated $\mathbb{C}^4/\pm 1$ singularities.

Lemma: If $C = C_1 \cup C_2$ with $C_1.C_2 = 2k$ then a neighbourhood of $[\mathcal{F}_1 \oplus \mathcal{F}_2] \in \mathrm{Prym}(\mathcal{C}/\mathcal{D})$ looks locally like $\mathbb{C}^{N-2k} \times (\mathbb{C}^{2k}/\pm 1)$.

Principally polarized examples

2b. Let S be a Kummer K3 surface with an elliptic fibration

$$S \longrightarrow E \times F / \pm 1 \longrightarrow E / \pm 1 \cong \mathbb{P}^1.$$

$\mathrm{Hilb}^2 S \to \mathbb{P}^2$ is an isotrivial fibration with smooth fibres $F \times F$.

The group $F[2] \cong \mathbb{Z}_2^{\oplus 2}$ acts by diagonal translation on $F \times F$ and fibrewise on $\mathrm{Hilb}^2 S$. Take the quotient $\mathrm{Hilb}^2 S / \mathbb{Z}_2^{\oplus 2}$ and blow-up codimension two singularities to get a symplectic orbifold X.

Prop: X is an isotrivial Lagrangian fibration over $\mathbb{P}^2$.

Rmk: Fibres $F \times F / F[2]$ are principally polarized. Moreover, X has $b_2 = 14$, $b_3 = 0$, $b_4 = 150$, and 36 isolated $\mathbb{C}^4 / \pm 1$ singularities.

Examples for $K_n(A)$

4a. Debarre system: Let $C \subset A$ give a polarization of type $(1, n+1)$. Then C has genus $n+2$ and $|C| \cong \mathbb{P}^n$. Let $\mathcal{C}/\mathbb{P}^n$ be the family of curves linearly equivalent to C and

$$Y := \overline{\mathrm{Jac}}^d(\mathcal{C}/\mathbb{P}^n) \longrightarrow \mathbb{P}^n.$$

We get a Lagrangian fibration

$$X := \mathsf{kernel}(\mathrm{Alb} : Y \longrightarrow A)$$

with $(1, \ldots, 1, n+1)$-polarized fibres $X_t = \mathsf{ker}(\overline{\mathrm{Jac}}^d C_t \longrightarrow A)$.

4b. If $A = E \times F$ then

$$\mathrm{Hilb}^{n+1} A \longrightarrow \mathrm{Sym}^{n+1} A \longrightarrow \mathrm{Sym}^{n+1} E \longrightarrow J^{n+1} E \cong E$$

induces an isotrivial Lagrangian fibration $K_n(A) \longrightarrow \mathbb{P}^n$ with $(1, \ldots, 1, n+1)$-polarized fibres

$$\cong \{(f_0, f_1, \ldots, f_n) \in F^{n+1} \mid f_0 + f_1 + \ldots + f_n = 0 \text{ in } F\}.$$

Summary of examples in four dimensions

Example	**Polarization type**
Beauville-Mukai system	$(1,1)$
$\mathrm{Hilb}^2 S$ of elliptic K3 S	$(1,1)$
Arbarello-Saccà-Ferretti system	$(1,1)$
Isotrivial on $\mathrm{Hilb}^2 S/\mathbb{Z}_2^{\oplus 2}$	$(1,1)$
Markushevich-Tikhomirov system	$(1,2)$
Isotrivial on $\mathrm{Hilb}^2 S/\mathbb{Z}_2$	$(1,2)$
Debarre system	$(1,3)$
Isotrivial on $K_2(A)$	$(1,3)$

Fibrations by Jacobians

Thm (Markushevich): Let $\mathcal{C}/\mathbb{P}^2$ be a flat family of integral Gorenstein curves of genus two such that $X = \overline{\mathrm{Jac}}^d(\mathcal{C}/\mathbb{P}^2)$ is a Lagrangian fibration (with X smooth!). Then $X \to \mathbb{P}^2$ must be a Beauville-Mukai integrable system.

Rmk: The general principally polarized abelian surface is the Jacobian of a genus two curve.

Thm (Matsushita): $R^i\pi_*\mathcal{O}_X \cong \Omega^i_{\mathbb{P}^2}$.

When $i = 1$ this says $TF \cong N^\vee_{F\subset X}$ for smooth fibres F.

Moreover, we have $R^1\pi_*\mathcal{O}_\mathcal{C} \cong R^1\pi_*\mathcal{O}_X \cong \Omega^1_{\mathbb{P}^2}$.

Fibrations by Jacobians

Proof: The relative canonical map gives a double cover

$$\mathcal{C} \longrightarrow \mathbb{P}(R^1\pi_*\mathcal{O}_{\mathcal{C}}) = \mathbb{P}(R^1\pi_*\mathcal{O}_X) = \mathbb{P}(\Omega^1_{\mathbb{P}^2}) \subset \mathbb{P}^2 \times (\mathbb{P}^2)^\vee$$

branched over the zero locus in $\mathbb{P}(\Omega^1_{\mathbb{P}^2})$ of a section of

$$\mathcal{O}_{\mathbb{P}(\Omega^1)}(6) \otimes \pi^*\mathcal{O}_{\mathbb{P}^2}(2k) = \mathcal{O}(2k+6,6)|_{\mathbb{P}(\Omega^1)}.$$

Now $R^1\pi_*\mathcal{O}_{\mathcal{C}} = \Omega^1_{\mathbb{P}^2}$ determines $k=-3$, so

$$\mathcal{O}(2k+6,6)|_{\mathbb{P}(\Omega^1)} = \mathcal{O}(0,6)|_{\mathbb{P}(\Omega^1)}$$

is pulled back from $(\mathbb{P}^2)^\vee$. This means the curves lie in the double cover of $(\mathbb{P}^2)^\vee$ branched over a sextic, i.e., a K3 surface.

$\square$

Fibrations by products of elliptic curves

Thm (Kamenova): Let $X \to \mathbb{P}^2$ be a Lagrangian fibration with

- X smooth,
- general fibre a product of two elliptic curves,
- "generic" singular fibres,
- and a global section.

Then X is birational to $\mathrm{Hilb}^2 S$ of an elliptic K3 surface S.

Rmk: Markushevich and Kamenova's theorems cover examples 1a and 1b. Next consider example 3a with $(1,2)$-polarized fibres.

Fibrations by $(1,2)$-polarized fibres

If A is $(1,2)$-polarized then $A^\vee$ is too. Let $C \subset A^\vee$ be a polarization. Then C is genus three, and pull-back gives

$$A = \mathrm{Pic}^0 A^\vee \longrightarrow \mathrm{Jac}^0 C \longrightarrow E,$$

i.e., A is the Prym variety of a double cover $C \to E$.

Thm (Qin-S-): Let $\mathcal{C}/\mathcal{E}/\mathbb{P}^2$ be a flat family of double covers of reduced Gorenstein curves of genus three and one, respectively, such that $X = \overline{\mathrm{Prym}}(\mathcal{C}/\mathcal{E})$ is a Lagrangian fibration. Then $X \to \mathbb{P}^2$ must be a Markushevich-Tikhomirov system

Thus the elliptic curves $\mathcal{E}$ must lie in a degree two del Pezzo and the genus three curves $\mathcal{C}$ must lie in its K3 double cover.

Fibrations by $(1,2)$-polarized fibres

Proof: $f : \mathcal{C} \to \mathcal{E}$ is branched over a divisor of degree four on each fibre. Thus there is a line bundle $\mathcal{L}$ of degree two on each fibre with

$$f_*\mathcal{O}_\mathcal{C} = \mathcal{O}_\mathcal{E} \oplus \mathcal{L}^\vee.$$

Applying $R^1\pi_*$ gives (π always denotes projection to $\mathbb{P}^2$)

$$R^1\pi_*\mathcal{O}_\mathcal{C} = R^1\pi_*\mathcal{O}_\mathcal{E} \oplus R^1\pi_*\mathcal{L}^\vee.$$

On the other hand, $\mathrm{Jac}^0 C \sim \mathrm{Jac}^0 E \times \mathrm{Prym}(C/E)$ implies

$$\mathrm{H}^1(C, \mathcal{O}_C) = \mathrm{H}^1(E, \mathcal{O}_E) \oplus \mathrm{H}^1(X_t, \mathcal{O}_{X_t}),$$

$$R^1\pi_*\mathcal{O}_\mathcal{C} = R^1\pi_*\mathcal{O}_\mathcal{E} \oplus R^1\pi_*\mathcal{O}_X.$$

Fibrations by $(1,2)$-polarized fibres

Therefore

$$R^1\pi_*\mathcal{L}^\vee \cong R^1\pi_*\mathcal{O}_X \cong \Omega^1_{\mathbb{P}^2},$$

and we have a double cover

$$h : \mathcal{E} \longrightarrow \mathbb{P}(R^1\pi_*\mathcal{L}^\vee) = \mathbb{P}(\Omega^1_{\mathbb{P}^2}) \subset \mathbb{P}^2 \times (\mathbb{P}^2)^\vee$$

branched over the zero locus in $\mathbb{P}(\Omega^1_{\mathbb{P}^2})$ of a section of

$$\mathcal{O}_{\mathbb{P}(\Omega^1)}(4) \otimes \pi^*\mathcal{O}_{\mathbb{P}^2}(2d) = \mathcal{O}(2d+4,4)|_{\mathbb{P}(\Omega^1)}.$$

Recall $\mathcal{C} \to \mathcal{E}$ is branched over the zero locus in $\mathcal{E}$ of a section of

$$\mathcal{L}^2 \cong h^*(\mathcal{O}_{\mathbb{P}(\Omega^1)}(2) \otimes \pi^*\mathcal{O}_{\mathbb{P}^2}(e)) = h^*\mathcal{O}(e+2,2)|_{\mathbb{P}(\Omega^1)}.$$

Now $R^1\pi_*\mathcal{L}^\vee = \Omega^1_{\mathbb{P}^2}$ determines $d=-2$ and $e=-2$.

Fibrations by $(1,2)$-polarized fibres

So

$$\mathcal{O}(2d+4,4)|_{\mathbb{P}(\Omega^1)} = \mathcal{O}(0,4)|_{\mathbb{P}(\Omega^1)},$$
$$h^*\mathcal{O}(e+2,2)|_{\mathbb{P}(\Omega^1)} = h^*\mathcal{O}(0,2)|_{\mathbb{P}(\Omega^1)}$$

are pulled back from $(\mathbb{P}^2)^\vee$.

This means the elliptic curves $\mathcal{E}$ lie in the double cover of $(\mathbb{P}^2)^\vee$ branched over a quartic, i.e., a degree two del Pezzo surface T.

Rmk: If $g : T \to (\mathbb{P}^2)^\vee$ then

$$K_T \cong g^*(\mathcal{O}(-3) \otimes \mathcal{O}(2)) = g^*\mathcal{O}(-1).$$

Moreover, the genus three curves $\mathcal{C}$ lie in the double cover of T branched over the pull-back of a conic $\cong K_T^{-2}$, i.e., a K3 surface. □

An invariant of Lagrangian fibrations

Thm (Wieneck): In a family of Lagrangian fibrations the polarization type of the fibres is constant.

For fibrations on $\mathrm{Hilb}^2 K3$ the polarization is *principal*

$$(d_1, d_2) = (1, 1).$$

For fibrations on $K_2(A)$ the polarization is of type

$$(d_1, d_2) = (1, 3).$$

Thm (Markman): If X is a general deformation of $\mathrm{Hilb}^2 K3$ admitting a Lagrangian fibration then it is birational to a *Tate-Shafarevich twist* of Beauville-Mukai system.

Finiteness

Thm (S-): Fixing $d_1 | \dots | d_n$, there are finitely many Lagrangian fibrations up to deformation with

- polarization type $(d_1, \dots, d_n)$,
- a global section,
- maximally varying fibres,
- semistable singular fibres in codimension one.

Rmk:

- *we want to bound the polarization type*,
- one expects finite Tate-Shafarevich twists up to deformation,
- van Geemen-Voisin and Bakker proved that Lagrangian fibrations are maximally varying or isotrivial,
- singular fibres are "almost" semistable after deforming.

Restrictions on polarization type

Thm (S-): Let $X \to \mathbb{P}^2$ be a Lagrangian fibration (with X smooth!) and let $Y \in \mathrm{H}^2(X, \mathbb{Z})$ restrict to a polarization of type (d_1, d_2) on each smooth fibre. Then $d_1 d_2$ can take only the following twenty values up to a square:

$$1, 2, 3, 5, 7, 10, 15, 61, 62, 241, 246, 247,$$

$$249, 251, 253, 254, 255, 257, 258, 259$$

Example: Polarization types $(1, 6)$ and $(1, 11)$ are *not* possible.

Rmk: In higher dimensions, the overall degree $d_1 \cdots d_n$ of the fibres can take only finitely many values up to an n^{th} power.

Restrictions on polarization type

Proof: Let L be the pullback of a hyperplane from $\mathbb{P}^2$. Then

$$\left(\int_X Y^2L^2\right)\left(\int_X c_2(\sigma\bar{\sigma})\right)^2 = \left(\int_X (\sigma\bar{\sigma})^2\right)\left(\int_X c_2 YL\right)^2$$

where $\int_X Y^2L^2 = 2!d_1d_2$ and Hitchin-S- formula gives

$$\frac{\left(\int_X c_2(\sigma\bar{\sigma})\right)^2}{\int_X(\sigma\bar{\sigma})^2} = \frac{24^2(2!)^2}{2^2}\sqrt{\hat{A}}[X].$$

Thus

$$1152d_1d_2\sqrt{\hat{A}}[X] = \left(\int_X c_2 YL\right)^2 \text{ is a square.}$$

Guan showed that $\sqrt{\hat{A}}[X]$ takes finitely many values.

$\square$

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