Lagrangian fibrations by Prym surfaces

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Outline:

- introduction to holomorphic symplectic geometry
- Lagrangian fibrations
- Prym varieties
- Beauville-Mukai systems
- Markushevich-Tikhomirov systems
- something new?
- Arbarello-Saccà-Ferretti systems

Irreducible holomorphic symplectic manifolds

Let M be a compact Kähler manifold with $c_1 = 0$.

Thm (Bogomolov): $\exists$ finite étale cover $\widetilde{M}$ of M with

$$\widetilde{M} = T \times \prod_i CY_i \times \prod_j IHS_j,$$

T = torus, CY_i = (strict) Calabi-Yau manifolds, and $IHS_j = \ldots$

Def: A compact Kähler manifold X is a *holomorphic symplectic manifold* if it admits a non-degenerate holomorphic two-form σ.

In addition if $\pi_1(X) = 0$ and $\mathrm{H}^0(\Omega^2)$ is generated by σ then we say X is an *irreducible holomorphic symplectic (IHS) manifold*.

Holomorphic symplectic varieties

Def (Beauville): X is a holomorphic symplectic variety if it is normal and X_{reg} admits a symplectic two-form that extends to a holomorphic two-form on any resolution of singularities.

Rmk: A symplectic desingularization may or may not exist.

On $\mathbb{C}^2/\pm 1$ the two-form $\sigma = dx \wedge dy$ extends to a non-degenerate two form on the blow-up

$$dx \wedge dy = dx \wedge d(\zeta x) = x dx \wedge d\zeta = \frac{1}{2} d(x^2) \wedge d\zeta$$

whereas on $\mathbb{C}^4/\pm 1$ the two-form will degenerate along the exceptional locus.

Examples of IHS manifolds

1. K3 surfaces S, e.g., double cover of $\mathbb{P}^2$ branched over a sextic, quartic in $\mathbb{P}^3$, hypersurface of bidegree $(2,3)$ in $\mathbb{P}^1 \times \mathbb{P}^2$.

2. Hilbert schemes of points on K3 surfaces, $\mathrm{Hilb}^n S \to \mathrm{Sym}^n S$.

3. Generalized Kummer varieties, $\widetilde{\mathrm{Hilb}^{n+1}} A = A \times K_n(A)$.
Equivalently $K_n(A) := \text{kernel}(\mathrm{Hilb}^{n+1} A \longrightarrow \mathrm{Sym}^{n+1} A \longrightarrow A)$.

4. Mukai moduli spaces of stable sheaves on K3/abelian surfaces.

$$\mathrm{Ext}^1(\mathcal{E},\mathcal{E}) \times \mathrm{Ext}^1(\mathcal{E},\mathcal{E}) \to \mathrm{Ext}^2(\mathcal{E},\mathcal{E}) \xrightarrow{\mathrm{tr}} \mathrm{H}^2(\mathcal{O}) \cong \mathbb{C}$$

Moduli of semistable sheaves are symplectic varieties.

5. O'Grady's spaces, OG6 and OG10.

Structure of fibrations

Let X be an IHS manifold of dimension $2n$.

Def: A submanifold $\iota : Y \hookrightarrow X$ is (holomorphic) Lagrangian if $\dim Y = n$ and $\iota^*\sigma$ vanishes identically.

Example: If $Y \cong \mathbb{P}^n$ then Y is Lagrangian, as $\mathrm{H}^0(\mathbb{P}^n, \Omega^2) = 0$.

Thm (Matsushita): Let $X \to B$ be a proper fibration, with connected fibres and $0 < \dim B < 2n$. Then

1. $\dim B = n = \dim F$,
2. F is Lagrangian wrt the holomorphic symplectic form σ,
3. generic fibre is a complex torus.

Structure of fibrations

Rmk: Hodge theory $\implies$ general fibre is an abelian variety.

Thm (Hwang): B is isomorphic to $\mathbb{P}^n$ if it is smooth.

Thm (Huybrechts-Xu): B is smooth if $n = 2$, thus $B \cong \mathbb{P}^2$.

We will call such a fibration $\pi : X \to \mathbb{P}^n$ a *Lagrangian fibration*.

Rmk: Lagrangian means $TF \subset TX$ is maximal isotropic wrt σ. Integrable means $T^*B \subset T^*X$ is maximal isotropic wrt σ^{-1}. Thus Lagrangian fibrations are equivalent to *integrable systems*.

Polarizations of abelian varieties

A polarization H of an abelian variety gives

$$c_1(H) \in \mathrm{H}^2(A, \mathbb{Z}) = \Lambda^2 \mathrm{H}_1(A, \mathbb{Z})^*.$$

With respect to a standard basis

$$c_1(H) = \begin{bmatrix} & & & & d_1 & & & \\ & & & & & d_2 & & \\ & & 0 & & & & \ddots & \\ & & & & & & & d_n \\ -d_1 & & & & & & & \\ & -d_2 & & & & & & \\ & & \ddots & & & & 0 & \\ & & & -d_n & & & & \end{bmatrix}$$

with $d_1 | d_2 | \cdots | d_n$. We call this the *type* of the polarization.

Elliptic K3 surfaces

Let S be a hypersurface of bidegree $(2, 3)$ in $\mathbb{P}^1 \times \mathbb{P}^2$. Projection to $\mathbb{P}^1$

$$S \longrightarrow \mathbb{P}^1$$

has fibres that are plane cubics, i.e., elliptic curves E.

Taking the Hilbert schemes gives a Lagrangian fibration

$$X := \mathrm{Hilb}^n S \longrightarrow \mathrm{Sym}^n S \longrightarrow \mathrm{Sym}^n \mathbb{P}^1 = \mathbb{P}^n$$

with general smooth fibre $E_1 \times E_2 \times \ldots \times E_n$. These are principally polarized, i.e., type $(1, 1, \ldots, 1)$.

(Generalized) Prym varieties

Let $\pi : C \to D$ be a double cover of curves with covering involution τ. Then

$$\mathrm{Fix}^0(\tau^*) = \pi^* \mathrm{Jac}^0 D \subset \mathrm{Jac}^0 C.$$

Def: The Prym variety of C/D is the abelian variety

$$\mathrm{Prym}(C/D) := \mathrm{Fix}^0(-\tau^*),$$

of dimension $g_C - g_D$, principally polarized if $\pi : C \to D$ has zero or two branch points, otherwise polarization type

$$(\underbrace{1, \ldots, 1}_{g_C - 2g_D}, \underbrace{2, \ldots, 2}_{g_D}).$$

Prym surfaces

To obtain a surface we need $g_C - g_D = 2$. Together with

$$2g_C - 2 = 2(2g_D - 2) + \# \text{ branch points},$$

this implies

$$2g_D + \# \text{ branch points} = 6.$$

g_C	g_D	**Branch points**	**Polarization type**
2	0	6	$(1,1)$
3	1	4	$(1,2)$
4	2	2	$(2,2)$
5	3	0	$(2,2)$

Goal: Construct/classify Lagrangian fibrations over $\mathbb{P}^2$ whose generic fibres are the above Prym surfaces.

The Beauville-Mukai integrable system

If C is a smooth genus n curve in a K3 surface S then $|C| \cong \mathbb{P}^n$.

Thm (Beauville-Mukai): Assume $\mathrm{NS}(S) \cong \mathbb{Z}.[C]$, so that all of the curves in the family $\mathcal{C}/\mathbb{P}^n$ linearly equivalent to C are reduced and irreducible. Then

$$X := \overline{\mathrm{Jac}}^d(\mathcal{C}/\mathbb{P}^n) \longrightarrow \mathbb{P}^n$$

is a Lagrangian fibration.

Rmk: By identifying X with the moduli space $M(0, [C], 1-n+d)$ of stable sheaves on S, we see that X is a deformation of $\mathrm{Hilb}^n S$.

Fibrations by Jacobian surfaces

Thm (Markushevich): Let $\mathcal{C}/\mathbb{P}^2$ be a flat family of reduced and irreducible Gorenstein curves of genus two such that $X = \overline{\mathrm{Jac}}^d(\mathcal{C}/\mathbb{P}^2)$ is a Lagrangian fibration. Then $X \to \mathbb{P}^2$ must be a Beauville-Mukai integrable system.

Proof: Matsushita proved that $R^i\pi_*\mathcal{O}_X \cong \Omega^i_{\mathbb{P}^2}$.

Moreover, we have $R^1\pi_*\mathcal{O}_{\mathcal{C}} \cong R^1\pi_*\mathcal{O}_X \cong \Omega^1_{\mathbb{P}^2}$.

Fibrations by Jacobian surfaces

The relative canonical map gives a double cover

$$\mathcal{C} \longrightarrow \mathbb{P}(R^1\pi_*\mathcal{O}_{\mathcal{C}}) = \mathbb{P}(R^1\pi_*\mathcal{O}_X) = \mathbb{P}(\Omega^1_{\mathbb{P}^2}) \subset \mathbb{P}^2 \times (\mathbb{P}^2)^\vee$$

branched over the zero locus in $\mathbb{P}(\Omega^1_{\mathbb{P}^2})$ of a section of

$$\mathcal{O}_{\mathbb{P}(\Omega^1)}(6) \otimes \pi^*\mathcal{O}_{\mathbb{P}^2}(2k) = \mathcal{O}(2k+6,6)|_{\mathbb{P}(\Omega^1)}.$$

Now $R^1\pi_*\mathcal{O}_{\mathcal{C}} = \Omega^1_{\mathbb{P}^2}$ determines $k = -3$, so

$$\mathcal{O}(2k+6,6)|_{\mathbb{P}(\Omega^1)} = \mathcal{O}(0,6)|_{\mathbb{P}(\Omega^1)}$$

is pulled back from $(\mathbb{P}^2)^\vee$. This means the curves lie in the double cover of $(\mathbb{P}^2)^\vee$ branched over a sextic, i.e., a K3 surface.

□

Families of Prym varieties

Let $\pi : S \to T$ be a K3 double cover of another surface with *anti-symplectic* covering involution τ.

Thm (Nikulin): There exist 75 anti-symplectic involutions τ on K3s. The quotient $T = S/\tau$ is an Enriques or a rational surface.

A curve $D \subset T$ has a double cover $C \subset S$,

$$\begin{array}{ccc} C & \subset & S \\ {\scriptstyle 2:1}\downarrow & & {\scriptstyle 2:1}\downarrow \\ D & \subset & T. \end{array}$$

Let $\mathcal{D} \to |D|$ be the complete linear system in T, let $\widetilde{\mathcal{C}} \to |C|$ be the complete linear system in S, and let

$$\mathcal{C} := \pi^* \mathcal{D} \subset \widetilde{\mathcal{C}}.$$

Families of Prym varieties

There are two commuting *anti-symplectic* involutions on the Beauville-Mukai system $\overline{\mathrm{Jac}}^0(\widetilde{\mathcal{C}}/|C|)$:

- the involution τ^* induced by τ,
- fibrewise duality $\mathcal{E} \mapsto \mathcal{E}xt^1_S(\mathcal{E}, \mathcal{O}(-C))$ (takes $\iota_*\mathcal{L} \mapsto \iota_*\mathcal{L}^\vee$).

Their composition is a symplectic involution.

Thm (Markushevich-Tikhomirov, Arbarello-Saccà-Ferretti, Matteini, Brakkee-Camere-Grossi-Pertusi-Saccà-Viktorova): We can construct a relative Prym variety

$$\mathrm{Prym}(\mathcal{C}/\mathcal{D}) := \mathrm{Fix}^0(\mathcal{E} \mapsto \mathcal{E}xt^1_S(\tau^*\mathcal{E}, \mathcal{O}(-C))) \subset \overline{\mathrm{Jac}}^0(\widetilde{\mathcal{C}}/|C|).$$

This is a symplectic *variety* and a Lagrangian fibration over $|D|$.

The Markushevich-Tikhomirov integrable system

Let S/T be a K3 double cover of a degree two del Pezzo, C/D a genus three cover of an elliptic curve, $\mathrm{Prym}(C/D)$ an abelian surface of polarization type $(1,2)$.

Thm (Markushevich-Tikhomirov): $\mathrm{Prym}(\mathcal{C}/\mathcal{D}) \to \mathbb{P}^2$ is an *irreducible* symplectic orbifold of dimension four, with 28 isolated $\mathbb{C}^4/\pm 1$ singularities.

Rmk: This orbifold is birational to the quotient of $\mathrm{Hilb}^2 S$ by a symplectic involution, sometimes called the *Nikulin variety*. Indeed the (relative) Abel-Prym map

$$\begin{array}{ccc} \mathrm{Hilb}^2 S & \dashrightarrow & \mathrm{Prym}(\mathcal{C}/\mathcal{D}) \\ \{p,q\} & \longmapsto & \mathcal{O}_C(p+q-\tau(p)-\tau(q)) \end{array}$$

is generically 2-to-1.

Fibrations by $(1,2)$-polarized surfaces

Thm (Qin-S-): Let $\mathcal{C}/\mathcal{E}/\mathbb{P}^2$ be a flat family of double covers of reduced Gorenstein curves of genus three and one, respectively, such that $X = \overline{\mathrm{Prym}}(\mathcal{C}/\mathcal{E})$ is a Lagrangian fibration. Then $X \to \mathbb{P}^2$ must be a Markushevich-Tikhomirov system

Thm (S-Teszler): The Markushevich-Tikhomirov system can be specialized to a fibration that is birational to a quotient of a Beauville-Mukai system by a symplectic fibre-preserving involution.

Genus four curves covering genus two curves

In the K3 surface S we have $C^2 = 2g_C - 2 = 6$.

Thus in T we have $g_D = 2$, $D^2 = 3$, and a triple cover

$$T \longrightarrow |D|^\vee = (\mathbb{P}^2)^\vee.$$

Thm (Miranda): T is isomorphic to $\mathbb{P}^2$ blown up at 10 points, with

$$D \in |4H - 2E_1 - E_2 - \ldots - E_{10}|.$$

Problem: Generically

$$|K_T^{-2}| = |6H - 2E_1 - \ldots - 2E_{10}|$$

is empty. We need to find special configurations of the 10 points.

Genus four curves covering genus two curves

Take a rational normal curve in $\mathbb{P}^6$ and project to a general plane. This gives the required plane sextic with 10 nodes.

Let S be the K3 double cover of T branched over this curve, C/D the genus four cover of a genus two curve.

Thm (S-): $\mathrm{Prym}(\mathcal{C}/\mathcal{D}) \to \mathbb{P}^2$ is a symplectic variety of dimension four, fibred by principally polarized abelian surfaces.

Qu: Is it irreducible? How many singularities? Are they resolvable?

Actually, all curves in $|D|$ appear to be irreducible, suggesting $\mathrm{Prym}(\mathcal{C}/\mathcal{D})$ is smooth, or at least has a smooth resolution!

Another example

Let p_1, p_2, p_3, p_4 and q_1, q_2, q_3, q_4 be points in $\mathbb{P}^1$, and blow up $\mathbb{P}^1 \times \mathbb{P}^1$ at the 9 points

$$(p_1, q_1), \qquad (p_1, q_2), \qquad (p_1, q_3), \qquad \ldots \qquad (p_3, q_3).$$

Then

$$|K_T^{-2}| = |4H_1 + 4H_2 - 2E_1 - \ldots - 2E_9| \neq \varnothing$$

as it contains the lines through p_1, p_2, p_3, p_4 and q_1, q_2, q_3, q_4. The double cover S is a Kummer K3 (with 7 A_1 singularities)

$$S = (E \widehat{\times F)/} \pm 1 \longrightarrow (E/ \pm \widehat{1) \times (}F/ \pm 1) = \mathbb{P}^1 \widehat{\times} \mathbb{P}^1,$$

where E and F are elliptic curves, double covers of $\mathbb{P}^1$ branched over p_1, p_2, p_3, p_4 and q_1, q_2, q_3, q_4.

Another example

Moreover

$$D \in |3H_1 + 2H_2 - E_1 - \ldots - E_9|.$$

must consist of lines through q_1, q_2, q_3 plus two additional lines through some p_4, p_5. Thus

$$|D| = \left\{\{p_4, p_5\} \subset \mathbb{P}^1\right\} = \mathrm{Sym}^2\mathbb{P}^1 = \mathbb{P}^2.$$

For the double cover C, the lines through q_1, q_2, q_3 are covered isomorphically by rational curves in S, whereas the lines through p_4, p_5 are covered by copies of F.

Generically

$$\mathrm{Prym}(C/D) \cong F \times F.$$

Another example

Thus in this case $\mathrm{Prym}(\mathcal{C}/\mathcal{D})$ is (birational to) the isotrivial Lagrangian fibration on $\mathrm{Hilb}^2 S$ with generic fibre $F \times F$ induced by

$$S \to (E \times F)/\pm 1 \to E/\pm 1 \cong \mathbb{P}^1.$$

Moreover, we can deform the earlier example to this one.

Conclusion: These integrable systems coming from genus 4/2 curves appear to be deformations of $\mathrm{Hilb}^2 S$.

Qu: Is the relative Abel-Prym map birational in this example?

$$\begin{array}{ccc} \mathrm{Hilb}^2 S & \dashrightarrow & \mathrm{Prym}(\mathcal{C}/\mathcal{D}) \\ \{p, q\} & \longmapsto & \mathcal{O}_C(p + q - \tau(p) - \tau(q)) \end{array}$$

The Arbarello-Saccà-Ferretti integrable system

Let S/T be a K3 double cover of an Enriques, C/D a genus $2n+1$ cover of a genus $n+1$ curve, $\mathrm{Prym}(C/D)$ principally polarized.

Thm (Arbarello-Saccà-Ferretti): $\mathrm{Prym}(\mathcal{C}/\mathcal{D}) \to \mathbb{P}^n$ is a symplectic variety, which is

- birational to a Beauville-Mukai system if D is hyperelliptic,
- simply connected with no symplectic resolution otherwise,
- and irreducible if n is even.

We are interested in the $n=2$ case. Then $\mathrm{Prym}(\mathcal{C}/\mathcal{D}) \to \mathbb{P}^2$ is an orbifold with isolated $\mathbb{C}^4/\pm 1$ singularities.

Genus five K3 surfaces covering Enriques

Let $S \subset \mathbb{P}^5$ be the intersection of three quadrics Q_i.

Lemma: If S admits a fixed-point free involution τ then

$$Q_i = A_i(x_0, x_1, x_2) + B_i(y_0, y_1, y_2)$$

and the involution is given by

$$[x_0, x_1, x_2, y_0, y_1, y_2] \mapsto [-x_0, -x_1, -x_2, y_0, y_1, y_2].$$

There are two $\mathbb{P}^2$-families of τ-invariant curves in S, given by intersection with hyperplanes

$$a_0x_0 + a_1x_1 + a_2x_2 = 0 \qquad \text{and} \qquad b_0y_0 + b_1y_1 + b_2y_2 = 0.$$

The Enriques surface

The Enriques surface is a 4-to-1 cover of the plane:

$$S \xrightarrow{2:1} T \xrightarrow{4:1} \mathbb{P}^2$$

The branch locus is a degree 12 curve $\Delta \subset \mathbb{P}^2$ with 36 cusps.

From this we can deduce that Δ has 162 bitangents, with 9 leading to reducible curves in T and S.

Lemma: $\mathrm{Prym}(\mathcal{C}/\mathcal{D})$ contains 36 isolated $\mathbb{C}^4/\pm 1$ singularities. It has Euler number

$$180 = 162 + 36 \times \frac{1}{2}.$$

Another (isotrivial) Lagrangian fibration

Let S be a Kummer K3 surface with an elliptic fibration

$$S \longrightarrow E \times F / \pm 1 \longrightarrow E / \pm 1 \cong \mathbb{P}^1.$$

$\mathrm{Hilb}^2 S \to \mathbb{P}^2$ is an isotrivial fibration with smooth fibres $F \times F$.

The group $F[2] \cong \mathbb{Z}_2^{\oplus 2}$ acts by diagonal translation on $F \times F$ and fibrewise on $\mathrm{Hilb}^2 S$. Take the quotient $\mathrm{Hilb}^2 S / \mathbb{Z}_2^{\oplus 2}$ and blow-up codimension two singularities to get a symplectic orbifold X.

Prop: X is an isotrivial Lagrangian fibration over $\mathbb{P}^2$.

Rmk: Fibres $F \times F / F[2]$ are principally polarized. The Betti numbers of X are $b_1 = 0$, $b_2 = 14$, $b_3 = 0$, and $b_4 = 150$.

And X has 36 isolated $\mathbb{C}^4 / \pm 1$ singularities and Euler number 180.

Questions

Qu: Is the isotrivial fibration $X \to \mathbb{P}^2$ a deformation of the ASF system $\mathrm{Prym}(\mathcal{C}/\mathcal{D}) \to \mathbb{P}^2$ in dimension four? Both have

- fibres that are principally polarized,
- 36 isolated $\mathbb{C}^4/\pm 1$ singularities,
- Euler number 180.

Recall that X is a partial resolution of $\mathrm{Hilb}^2 S/\mathbb{Z}_2^{\oplus 2}$. We also have the relative Abel-Prym map

$$\begin{array}{ccc} \mathrm{Hilb}^2 S & \dashrightarrow & \mathrm{Prym}(\mathcal{C}/\mathcal{D}), \\ \{p,q\} & \longmapsto & \mathcal{O}_C(p+q-\tau(p)-\tau(q)). \end{array}$$

Prop (Casalaina-Martin): This map has generic degree four.

Qu: Is it a quotient by a symplectic action of $\mathbb{Z}_2^{\oplus 2}$?

Pryms as Jacobians

There is a net of quadrics containing the K3 surface $S \subset \mathbb{P}^5$. The singular quadrics are parametrized by two cubics in this net.

There is a net of quadrics containing a curve $C \subset \mathbb{P}^4$. The singular quadrics are parametrized by the union of a cubic and a conic.

Exer (Arbarello-Cornalba-Griffiths-Harris): $\mathrm{Prym}(C/D)$ is isomorphic to the Jacobian of the genus two curve that double covers the conic with branching at its intersection with the cubic.

So the ASF system is a family of Jacobians of genus two curves. However, Markushevich's Theorem does not apply as some of the genus two curves are reducible.

Summary

g_C	g_D	**Polarization type**	**Example**	**Status**
2	0	$(1,1)$	BM system	classified
3	1	$(1,2)$	MT system	classified
4	2	$(2,2)$	new?	constructed?
5	3	$(2,2)$	ASF system	constructed